Math Circles - Group Theory

Question Sheet 2

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1. (a) Draw out the multiplication tables for $(\text{Poly}(3), \ast)$, $(\text{Sym}(3), \ast)$. Compare them and write down anything you notice. Are they isomorphic?

(b) The two groups from part (a), $(Z_3, +)$, and $Z_2 \times Z_3$ all have order 6. Which of these four groups are isomorphic? Which one’s aren’t? Prove your answers.

2. Find all groups of order 4. How many different (i.e. not isomorphic) groups are there. Find all groups of order 5 (this is the most important question on this sheet!).

3. There is a copy of $(Z_3, +)$ in $(\text{Poly}(3), \ast)$. Where is it? There is a copy of $(\text{Poly}(n), \ast)$ in $(\text{Sym}(n), 

Where is it? Write down formally what it means to have a copy of one group in another.

4. Calculate the orders of all of the elements in $(\mathbb{Q}_8, \cdot)$.

5. Prove that inverses are unique. That is for any element $a$ in a group, if $ba = e = ca$, then $b = c$.

6. Prove that identities are unique. That is, if $e$ and $f$ are two elements such that $ea = a$ and $fa = a$ for all elements $a$ in the group, then $e = f$.

7. Find all subgroups of $(\mathbb{Q}_8, \cdot)$ and find their orders. Do the same for $(Z_n, +)$, and $(\text{Sym}(3), \ast)$. Any conjectures?

8. (a) i. In $(\text{Braid}(3), \ast)$, what is $\begin{array}{c|c|c} * & X & X \\ \hline X & ? & ? \\ \hline X & ? & ? \\ \end{array}$? What is the inverse of $\begin{array}{c|c} * & X \\ \hline X & ? \\ \end{array}$?

ii. In $(\text{Sym}(3), \ast)$, what is $\begin{array}{c|c|c} * & X & X \\ \hline X & ? & ? \\ \hline X & ? & ? \\ \end{array}$? What is the inverse of $\begin{array}{c|c} * & X \\ \hline X & ? \\ \end{array}$?

(b) i. In $(Z, +)$, what is $6 + 7$? What is the inverse of 8?

ii. In $(Z_9, +)$, what is $6 + 7$? What is the inverse of 8?

(c) Write down what you think the relationship between $(\text{Braid}(3), \ast)$ and $(\text{Sym}(3), \ast)$ is. Write down what you think the relationship between $(Z, +)$ and $(Z_9, +)$ is. What am I trying to hint at in this question?

9. A group $G$ has a generator if there is an element $a$ such that $|a| = |G|$. Which of the following groups has a generator? $(\mathbb{Z}_4, +)$, $(\mathbb{Z}_8^\ast)$, $(\{1, -1, i, -i\}, \times)$. Can you say anything about which of these groups are isomorphic? Which aren’t? Why would such an element be called a generator?

10. Look carefully at all the examples of groups you have played with so far, and keep a list of anything you think might be true about groups. Try and prove your guesses.