1. Determine if the following series converge or diverge, if the series converges find its sum.

(a) \( \sum_{n=0}^{\infty} \left( \frac{13}{12} \right)^n \).

**Solution:** This series diverges since \( r = \frac{13}{12} \geq 1 \).

(b) \( \sum_{n=0}^{\infty} \left( \frac{2}{7} \right)^{n+1} \)

**Solution:** This series converges, \( r = \frac{2}{7} < 1 \), and we get
\[
\sum_{n=0}^{\infty} \left( \frac{2}{7} \right)^{n+1} = \sum_{n=0}^{\infty} \frac{2}{7} \cdot \left( \frac{2}{7} \right)^n = \frac{2}{7} \cdot \frac{1}{1 - \frac{2}{7}} = \frac{2}{5}.
\]

(c) \( \sum_{n=0}^{\infty} 2^{3n} \cdot 5^{2-n} \).

**Solution:** Let’s simplify first:
\[
\sum_{n=0}^{\infty} 2^{3n} \cdot 5^{2-n} = \sum_{n=0}^{\infty} 2^{3n} \cdot 5^2 \cdot 5^{-n} = \sum_{n=0}^{\infty} 25 \cdot \frac{8^n}{5^n} = \sum_{n=0}^{\infty} 25 \cdot \left( \frac{8}{5} \right)^n
\]
and so this series diverges since \( r = \frac{8}{5} \geq 1 \).

(d) \( \sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+1}} \).

**Solution:**
\[
\sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{6} \cdot \frac{2^n}{3^n} = \sum_{n=0}^{\infty} \frac{1}{6} \cdot \left( \frac{2}{3} \right)^n = \frac{1}{6} \cdot \frac{1}{1 - \frac{2}{3}} = \frac{1}{2}.
\]
In this case \( r = \frac{2}{3} < 1 \).

(e) \( \sum_{n=1}^{\infty} \left( \frac{4}{9} \right)^n \)

**Solution:** Clearly the series converges since \( r = \frac{4}{9} < 1 \), but the index of this series doesn’t start at the usual \( n = 0 \). If \( n \) were zero then the first term would be 1, so let’s add and subtract that:
\[
\sum_{n=1}^{\infty} \left( \frac{4}{9} \right)^n = -1 + 1 + \sum_{n=1}^{\infty} \left( \frac{4}{9} \right)^n = -1 + \sum_{n=0}^{\infty} \left( \frac{4}{9} \right)^n = -1 + \frac{1}{1 - \frac{4}{9}} = -1 + \frac{9}{5} = \frac{4}{5}.
\]
2. Write the following infinite decimals as a single fraction

(a) 0.1919191919... = 0.19.

Solution:

\[ 0.19 = \frac{19}{100} + \frac{19}{100^2} + \cdots = \frac{19}{100} \left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots\right) = \frac{19}{100} \sum_{n=0}^{\infty} \left(\frac{1}{100}\right)^n \]

which is a convergent series (since \( r = \frac{1}{100} < 1 \)) that sums up to

\[ \frac{19}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{19}{99}. \]

(b) 1.123412341234... = 1.1234.

Solution:

\[ 1.1234 = 1 + \frac{1234}{10000} + \frac{1234}{10000^2} + \cdots = 1 + \frac{1234}{10000} \left(1 + \frac{1}{10000} + \frac{1}{10000^2} + \cdots\right) \]

\[ = 1 + \frac{1234}{10000} \sum_{n=0}^{\infty} \left(\frac{1}{10000}\right)^n. \]

Again, this series converges to

\[ 1 + \frac{1234}{10000} \cdot \frac{1}{1 - \frac{1}{10000}} = 1 + \frac{1234}{9999} = \frac{11233}{9999}. \]

3. Suppose you tweak the defense shields around the planet so that they now reflect \( \frac{1}{3} \) of the beam, absorb \( \frac{5}{9} \) of the beam and transmit \( \frac{1}{9} \) of the beam. What fraction of the starting intensity (I) gets through now?

Solution: The set up and picture are similar to the one in the notes. But in this case the total beam intensity that gets through is

\[ \frac{I}{81} + \frac{I}{9(81)} + \frac{I}{9^2(81)} + \cdots = \frac{I}{81} \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n = \frac{I}{81} \cdot \frac{1}{1 - \frac{1}{9}} = \frac{I}{81} \cdot \frac{9}{8} = \frac{I}{72}. \]

That’s better than before!

4. Prove that \( \sum_{n=1}^{\infty} \left[ (-1)^n \left(\frac{2}{n}\right) \cos^2 \left(\frac{n\pi}{2}\right) \right] \) diverges.

Solution: This series looks ridiculous! But let’s examine each part. The \((-1)^n\) term alternates between \(-1\) if \(n\) is odd and \(1\) if \(n\) is even. The \(\frac{2}{n}\) term looks like twice the harmonic series.
Finally, what does \( \cos^2 \left( \frac{n\pi}{2} \right) \) look like? If \( n \) is even, say \( n = 2k \), then we get \( \cos^2(k\pi) = 1 \). On the other hand, if \( n \) is odd then \( \cos^2 \left( \frac{n\pi}{2} \right) = 0 \). Therefore, the overall term is zero if \( n \) is odd! So let’s only consider the even \( n \), let \( n = 2k \) and we get

\[
\sum_{n=1}^{\infty} \left[ (-1)^n \left( \frac{2}{n} \right) \cos^2 \left( \frac{n\pi}{2} \right) \right] = \sum_{k=1}^{\infty} \left[ (-1)^{2k} \left( \frac{2}{2k} \right) \cos^2 (k\pi) \right] = \sum_{k=1}^{\infty} \frac{1}{k}.
\]

But that’s just the harmonic series! We’ve already proven that it diverges in the lecture.

5. Find a simple expression for the \( n \)th partial sum of the series and use it to find the sum of the series:

\[
\sum_{n=0}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]

Such series are called telescoping series and are another example of a series in which we can actually find the sum.

**Solution:** Let’s start looking at the partial sums.

\[
S_1 = \frac{1}{1} - \frac{1}{2},
\]

\[
S_2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{1}{1} - \frac{1}{3},
\]

\[
S_3 = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = \frac{1}{1} - \frac{1}{4},
\]

and so on. Now we can see the pattern:

\[
S_n = 1 - \frac{1}{n+1}.
\]

What happens to \( S_n \) as \( n \to \infty \)? Clearly \( \frac{1}{n+1} \to 0 \) as \( n \to \infty \), so \( S_n \to 1 \). This series adds up to 1!

6. What’s wrong with the following argument?

Let \( x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots \) and \( y = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots \). Then clearly \( 2y = x + y \) which means \( x = y \). But notice that

\[
x - y = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{5} - \frac{1}{6} \right) + \cdots > 0
\]

since it’s a sum of positive numbers. Therefore \( x > y \). So we have shown that \( x = y \) and \( x > y \).

**Solution:** The problem is actually right at the start. Both \( x \) and \( y \) are infinite! Each is some part of the harmonic series (\( x \) has the odd terms and \( y \) has the even terms). Therefore it doesn’t make sense to talk about their values nor do arithmetic with them!