UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

Winter 2015 - March 10/11

Number Theory

Introduction

Today, we will be looking at some properties of numbers - known as **number theory**. More specifically, we will learn about palindromes and triangular numbers, before looking at prime numbers.

Palindromic Numbers

What do you notice about the following images?



These images are symmetrical. However, it is not only pictures that can follow this property: words and numbers can as well.

A **palindrome** is a word or phrase that reads the same from left to right. Examples of palindromes include Madam, Hannah, and Go dog.

We call a number a **palindromic number** if it reads the same from the left as it does from the right. Just like these images, palindromic numbers are symmetrical: If you draw a line through the middle of the number (a vertical line, in the case of a number), you will get the same number on either side.

Some examples of palindromes are 1331, 404, 9, 77777, and 145686541. Make sure they read the same from the left and right!

Examples

At the Waterloo Marathon, everyone has a bib with a number on it. You are watching the runners going by and taking note of their bib number.

1. Dalmatian is the smallest 3-digit palindromic number. What number is Dalmatian?

101

2. Diana is the largest single-digit even palindromic number. What number is Diana?

8

3. The product of Taylor's two digits is 4. What palindromic number is Taylor?

22 4. Here comes Mel! He stops running to tell you about his palindromic bib number: The sum and product of its five digits are both 8. What number is Mel?

21212

5. Hannah is a six-digit palindromic number whose digits sum to 6. What are the possibilities of Hannah's number?

111111,102201,120021,210012,201102,300003

Finding Palindromes

One way to find a palindrome is as follows:

1. Pick any number
2. Reverse the digits of the number
3. Add these two numbers together
4. Repeat until you get a palindrome

For example, we can find a palindrome using the number 37 as follows:

37 reversed is 73.

37 + 73 = 110.

110 reversed is 11.

110 + 11 = 121, which is a palindrome.

1. Using the number 86, find a palindrome.

86 + 68 = 154

154 + 451 = 605

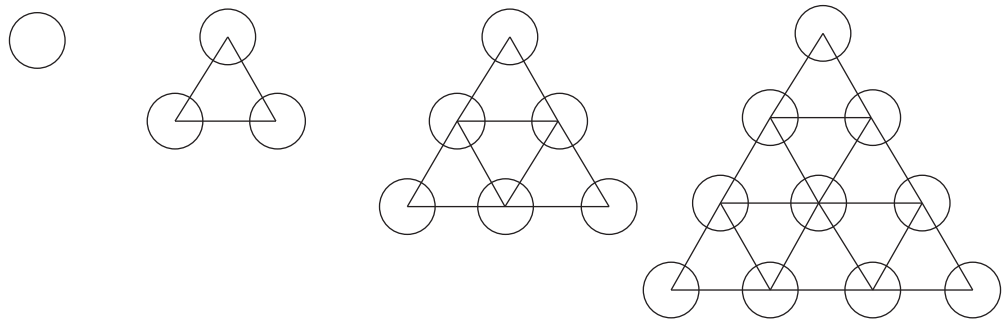
605 + 506 = 1111

2. Using the number 412, find a palindrome.

412 + 214 = 626

Triangular Numbers

Consider the following pattern:



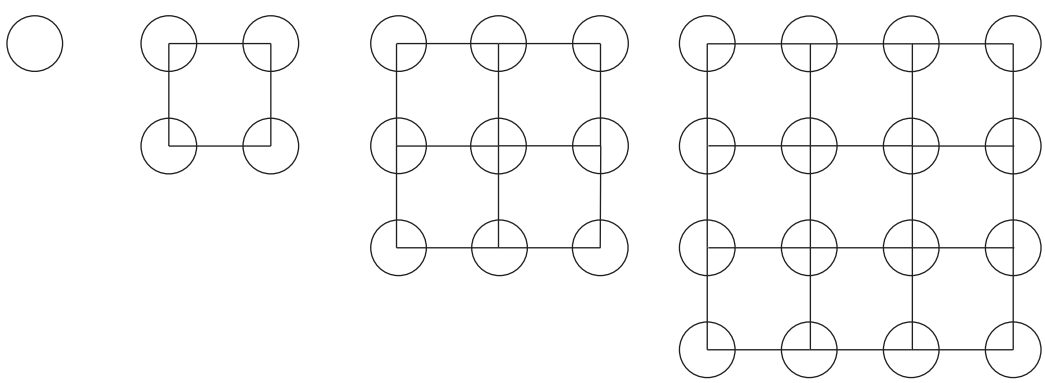
What is the rule of the pattern?

Add one to the number we are adding by each time; arrange the circles into a triangle of side length t_n-1

If a number of dots can be arranged into a regular (equilateral) triangle, we call that number a **triangular number**. The sequence of triangular numbers is {1,3,6,10,...}.

From this sequence, we can find that the formula for triangular numbers is $t_n = \dfrac{n(n+1)}{2}$, where t_n is the nth term.

But triangles aren't the only shape we can consider. In fact, we can consider any shape, so we could have any set of **polygonal numbers**. Let's consider the set of square numbers:



Notice that these numbers look familiar! We tend to know the square numbers since we know our square roots so well. We also call the set of square numbers **perfect squares** since their square roots are integers.

Examples

1. What is the 13th triangular number?

91

2. What is the 13th square number?

169

Prime Numbers

A **prime** number is a natural number that can only be divided by 1 and itself.

A **composite** number is a natural number that has more **factors** than just 1 and itself.

For example, 2, 3, and 5 are prime numbers, since $2 = 2 \times 1$ but there is no other way to multiply to get 2. 4, 6, and 9 are composite numbers since $4 = 2 \times 2$ and $4 = 4 \times 1$.

One method for finding the prime numbers is by using the **Sieve of Eratosthenes**. Here are the steps to this algorithm, using the following table:

1. Cross out 1 (it is not prime)
2. Circle 2 (it is prime) and then cross out all multiples of 2
3. Circle 3 (it is prime) and then cross out all multiples of 3
4. Circle 5, then cross out all multiples of 5
5. Circle 7, then cross out all multiples of 7
6. Continue by circling the next number not crossed out, then cross out all of its multiples

The circled numbers are all the prime numbers less than 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Prime Factorization

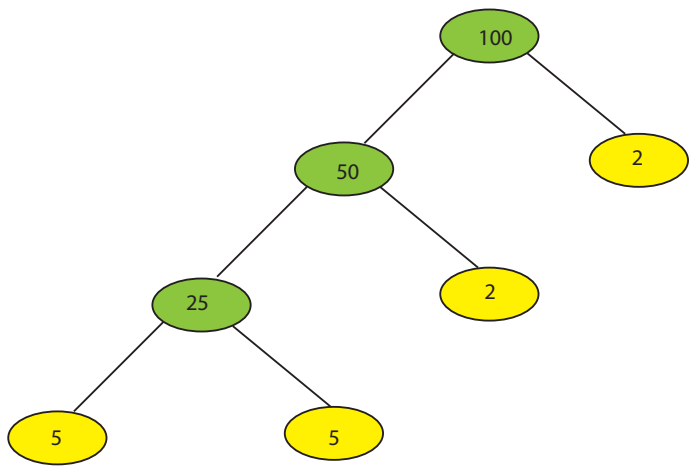
Any number can be expressed as the product of prime numbers.

For example, $100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$

To perform prime factorization on a number, simply find any of its factors. Then, break the factors down until they are prime numbers. For 100, we know that $100 = 50 \times 2$. 2 is prime, but $50 = 25 \times 2$. Again, 2 is prime, but $25 = 5 \times 5$. Since 5 is prime, we have the prime factorization of 100.

An easier way to perform prime factorization on a number is by using a **factor tree**, to break down the factors of a number into primes.

Here is the factor tree of 100:



Examples

Find the prime factorization of the following:

a) 21

3×7

b) 64

2^6

c) 75

$5^2 \times 3$

d) 97

97

Greatest Common Divisor

The **greatest common divisor** (GCD) of two numbers is the greatest number that is a factor of both numbers. For example, the GCD of 24 and 15 is 3.

You are probably familiar with one method of finding the GCD, where we list all the factors of both numbers, and the greatest number is be the GCD.

There is in fact an easier way to find the GCD of two numbers:

1. Find the prime factorizations of both numbers
2. Multiply the common prime factors together
3. The product is the GCD

For example, let's find the GCD of 24 and 36.

$24 = 2 \times 2 \times 2 \times 3$, and $36 = 2 \times 2 \times 3 \times 3$. The prime factorizations of 24 and 36 have one 3 and two 2's in common.

So, the GCD of 24 and 36 is $2 \times 2 \times 3 = 12$.

1. Find the GCD of 36 and 75.

3

2. Find the GCD of 105 and 84.

21

Least Common Multiple

We know that a **multiple** of a number is that number multiplied by any non-zero number. A **common multiple** of two numbers is a number that is a multiple of both numbers. So, the **least common multiple** (LCM) is the smallest number that is a multiple of both numbers.

You probably know one method of finding the LCM, where you list all the multiples, and the lowest number is the LCM. Again, there is a faster way to find the LCM using prime factorization:

1. Find the prime factorizations of both numbers
2. Multiply the prime factors the greatest number of times they appear in either number
3. The product is the LCM

For example, let's find the LCM of 9 and 12.

$9 = 3 \times 3$ and $12 = 2 \times 2 \times 3$.

3 appears twice in 9, which is greater than the one time it appears in 12.

2 appears twice in 12.

So, the LCM of 9 and 12 is $3 \times 3 \times 2 \times 2 = 36$.

1. Find the LCM of 7 and 15.

105

2. Find the LCM of 8 and 14.

56

Prime Factorization of Perfect Squares

Recall that the perfect squares are as follows: {1,4,9,16,25,...}

Find the prime factorization of the perfect squares.

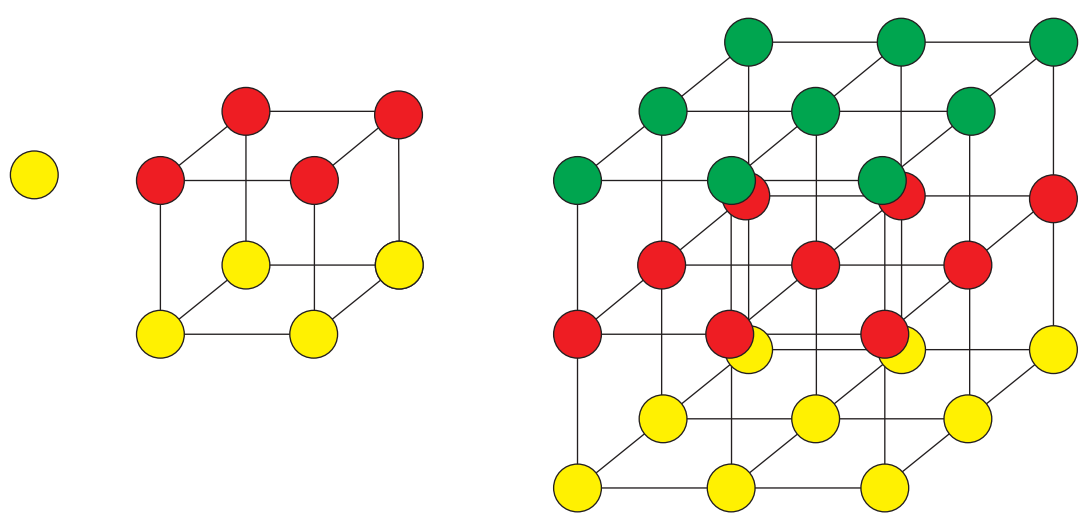
1, 2^2, 3^2, 2^4, 5^2,...

Notice that the **multiplicity** of each prime factor, or the number of times each prime factor appears, is an even number.

Is this a coincidence? To find out, we need to consider another set similar to perfect squares. Does any such set exist?

From our knowledge of exponents, we know that the **cube root** of a number x is a number that, when multiplied by itself three times (or cubed), gives x. For example, $2^3 = 8$, so 2 is the cube root of 8, written $\sqrt[3]{8} = 2$.

Since we have perfect squares, we also have **perfect cubes**.



The perfect cubes are {1,8,27,64,...}

Now find the prime factorization of the perfect cubes.

1, 2^3, 3^3, 2^6,...

We realize that the multiplicity of each of prime factors of the perfect cubes is a multiple of 3.

We will then conclude that the multiplicity of each prime factor of an nth root is a multiple of n.

Examples

1. If you know that 243 is a 5th root, find its prime factorization.

3^5

2. What is the multiplicity of the prime factors of 2187?

7: $2187 = 3^7$

Perfect Square Palindromes

Are any palindromes perfect squares? Let's take a look:

$11^2 = 121$

$101^2 = 10201$

$1001^2 = 1002001$

$10001^2 = 100020001$

What is the pattern of these perfect squares?

1{number of 0's equal to number of 0's in number}2{number of 0's equal to number of 0's in number}1

What is $1\,000\,000\,000\,000\,000\,001^2$?

$1\,000\,000\,000\,000\,000\,002\,000\,000\,000\,000\,000\,001$

What is $\sqrt{1\,000\,000\,002\,000\,000\,001}$?

$1\,000\,000\,001$

Perfect Cube Palindromes

Can we find a similar pattern with perfect cubes?

$11^3 = 1331$

$101^3 = 1030301$

$1001^3 = 1003003001$

What is the pattern of these perfect cubes?

1{number of 0's in number}3{number of 0's in number}3{number of 0's in number}1 What is $1\,000\,001^3$?

1 000 003 000 003 000 001

What is $\sqrt[3]{1\,000\,000\,003\,000\,000\,003\,000\,000\,001}$?

1 000 000 001

Problems

1. How many 1-digit palindromes are there?

2. Which of the following are palindromes?
a) HANNAHBANANABHANNAH
b) 1030201
c) ABCDECBA
d) 1771

3. Find a palindrome from the following numbers:
a) 18
b) 886
c) 3742

4. 2002 was the last palindromic year. What will be the next palindromic year?

5. How many 7-digit palindromic numbers are there?

6. How many 8-digit palindromic numbers are there?

7. How many palindromic numbers are there between the numbers 100 and 400?

8. Naomi is finishing the marathon and George is waiting for her at the finish line. He can't remember her bib number, just that it was the next palindromic number after his bib number, 5678. What bib number does Naomi have?

9. Sachin plays for the Waterloo quidditch team and made sure he picked out a jersey number that is a palindrome. Quidditch rules state that his jersey number must be less than 1000. Sachin chose an even number, where the sum of the digits in his number is 4. What is Sachin's jersey number?

10. Find the prime factorization of the following numbers:
a) 472
b) 87
c) 10500
d) 360

11. Find the GCD of the following pairs (or groups) of numbers:
a) 37 and 73
b) 50 and 200
c) 172 and 360
d) 328, 216 and 400

12. Find the LCM of the following pairs (or groups) of numbers:
a) 13 and 39
b) 42 and 24
c) 12 and 14
d) 15, 21 and 18

13. Given the expression $2^{2k}\ 3^{3k}\ 5^{5k} = 337\,500$, what is k?

14. Given the expression $\sqrt{2^k 3^{3k} 5^{5k}} = 168\,750$, what is k?

15. Given the expression $2^{2k}\ 3^{k}\ 5^{\ k} = 216\,000$, what is k?

16. What is the multiplicity of the prime factors of 9765625?

17. What is $1\,000\,000\,001^2$?

18. What is $\sqrt{100\,000\,020\,000\,001}$?

19. What is 10001^3?

20. What is $\sqrt[3]{1\,000\,300\,030\,001}$?

21. If the square of perfect fourths is similar to those of perfect cubes, where $101^4 = 104060401$, find $\sqrt[4]{10004000600040001}$.

22. Find 100001^4

23. What is the LCM of 18 and 14 that is also a perfect square?

Solutions

1. 9.

2. a) Yes
b) No
c) No
d) Yes

3. a) 99
b) 1136311
c) 25652

4. 2112

5. 9000

6. 9000

7. 30

8. 5775

9. 202

10. a) 59×2^3
b) 87 c) $3 \times 7 \times 5^3 \times 2^2$

11. a) 1
b) 50
c) 4
d) 8
12. a) 39
b) 168
c) 84
d) 630

13. k = 1

14. k = 2

15. k = 3

16. 10: 5^{10}

17. 1 000 000 002 000 000 001

18. 10 000 001

19. 1 000 300 030 001

20. 10 001

21. 10001

22. 100 004 000 060 000 400 001

23. $18 = 2 \times 3^2$

$14 = 2 \times 7$

But each prime factor must be square to get a perfect square LCM.

So the LCM is $2^2 \times 3^2 \times 7^2 = 1764$