UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

Winter 2015 - February 17/18

Sequences

What is a Sequence?

A **sequence** is an ordered list of items that follow a certain **rule**. Today, we will be looking at lists of numbers. We will find the rules of different sequences, while exploring some properties and special types of sequences. Let's start with some notation and definitions.

Notation

Sequences are very similar to sets, which we covered a few weeks ago. The main difference between the two is that while sets are unordered, the order of a sequence does matter. In fact, the order of the sequence is what defines its rule. The best way to explain this is through an example:

When we take the set of positive even numbers less than 10, we can write {2,4,6,8} or {4,8,2,6} - the order does not matter.

If we take the sequence {2,4,6,8}, the rule to this sequence could be: "Starting at 2, we add 2 to get to the next term." Or, the rule could be "Increasing positive even numbers less than 10." No matter which we choose, the order is important.

Another difference between sets and sequences is that we do not call the items in a sequence its elements any more. We say that an item in a sequence is a **term**. We will often ask you to find a certain term of a sequence, for example, the 3rd term, or the 20th term. We write t_3 for the 3rd term, and t_{20} for the 20th term. In general, $\mathbf{t}_n$ is the n^{th} term, where n is the term number.

We can also repeat terms in a sequence, something that we couldn't do with sets. We still use the curly brackets { } to show the terms of a sequence.

Let's try some examples.

1. Given the sequence {1,3,5,7,9,...}

What is a rule of this sequence?

Odd numbers beginning with 1

or

Beginning with 1, add 2 to get each term.

What is t_6?

11 2. Given the sequence {1,6,11,16,21,...}

What is the rule of this sequence?

Beginning with 1, add 5 to get the next term.

What is t_8?

36

The Closed-Form Definition of a Sequence

The **closed-form** definition of a sequence is used when we define a sequence based on the term number. For example, for the sequence {2,4,6,8,...}, we can say that $t_n = 2n$, since $t_1 = 2$, $t_2 = 4$, $t_3 = 6$, and we know that $1 \times 2 = 2$, $2 \times 2 = 4$, and $3 \times 2 = 6$. In words, we would say "multiply the term number by 2" or "double the term number."

Consider the sequence {1,3,5,7,9,...}. We know the following:

n	$\mathbf{t}_n$
1	1
2	3
3	5
4	7
5	9

What pattern do you notice between n and t_n?

We multiply the term number by 2, then subtract 1

When we write closed-form definitions, we include n in the definition.

In this case, we can write the following equation: $t_n = 2n - 1$.

The Recursive Definition of a Sequence

Instead of writing the term t_n with respect to the term number, we can write t_n based on the previous term. Let's consider the sequence {2,4,6,8,...} again. In words, the rule would be "Beginning at 2, add 2 to each term to get the next term." How can we write this as an equation based on the previous term?

Before we do this, let's introduce a new notation. When we want to call the previous term, we use $\mathbf{t}_{n-1}$.

We also need to incorporate the "Beginning at 2" into our equation. To do this, we will always state t_1 before the equation.

Returning to our example, we know that t_n = previous term + 2

So, the **recursive definition** of this sequence is as follows:

$t_1 = 2$

$t_n = t_{n-1} + 2$.

What is the recursive definition of the sequence {1,2,4,8,...}?

$t_1 = 1$

$t_n = t_{n-1} \times 2$

Practice: Closed-Form and Recursive Definitions

1. Describe the sequence using a closed-form definition.

a) {6,7,8,9,...}

$t_n = n + 5$ b) {3,6,9,12,...}

$t_n = 3n$ c) {1,4,9,16,...}

$t_n = n \times n = n^2$

2. Describe the sequence using a recursive definition. Name the initial term.

a) {20, 14, 8, 2, ...}

$t_1 = 20$

$t_n = t_{n-1} - 6$

b) {1,3,9,27,...}

$t_1 = 1$

$t_n = 3 \times t_{n-1}$

c) {2,4,16, 256,...}

$t_1 = 2$

$t_n = t_{n-1} \times t_{n-1} = t_{n-1}{}^2$

Now let's look at some recursive sequences that have special properties.

Arithmetic Sequences

Consider the sequence {1,4,7,10,...}.

We notice that this is a recursive sequence with the rule $t_n = t_{n-1} + 3$ and $t_1 = 1$.

That is, we add 3 to each term to get the next term.

An **arithmetic sequence** is a sequence where we add or subtract the same amount between each term. Algebraically, we can think of an arithmetic sequence as follows:

$\mathbf{t}_n = \mathbf{t}_{n-1} + \mathbf{d}$

where d is the **common difference**, or the number we are adding each time (d can be positive or negative!).

Find d and the next term in the arithmetic sequence:

a) {2,6,10,14,...}

$d = 4$

$t_5 = 18$

b) {20,19,18,17,...}

$d = -1$

$t_5 = 16$

Using arithmetic sequences, we can also find the n^{th} term in a series. That is, we can find the 20th term even if we are only given the first 4. Here is the formula for the n^{th} term:

$\mathbf{t_n = t_1 + [d \times (n\text{-}1)]}$

Find t_n and t_8 for the following sequences:

a) {1,6,11,16,...}

$t_n = 1 + 5(n-1)$

$t_8 = 36$

b) {30,27,24,21,...}

$t_n = 30 - 3(n-1)$

$t_8 = 9$

Geometric Sequences

Consider the sequence {1, 3, 9, 27,...}.

We notice that this is a recursive sequence with the rule $t_n = t_{n-1} \times 3$, with $t_1 = 1$. That is, we multiply each term by 3 to get the next term.

A **geometric sequence** is a sequence where we multiply each term by the same constant. The formula for geometric sequence is as follows:

$\mathbf{t_n = t_{n-1} \times r}$

where r is the **common ratio**, or the number that we are multiplying by each time. (Remember: multiplying by 0.5 is the same as dividing by 2!)

Find the common ratio and the next term in the sequence:

a) {2, 4, 8, 16, ...}

$r = 2$

$t_5 = 32$

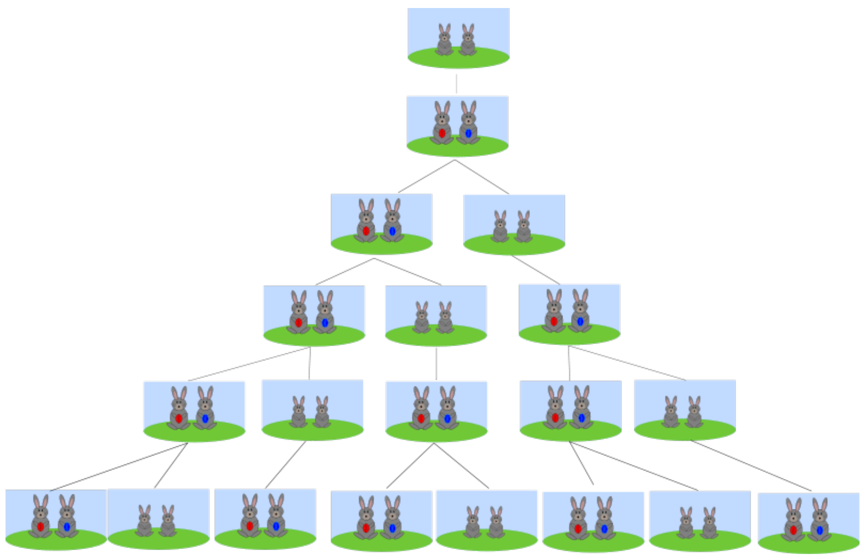
b) {2500, 500, 100, 20, ...}

$r = 0.5$

$t_5 = 10$

Fibonacci Sequence

Suppose a pair of rabbits, one male and one female, are put in a field to mate. The rabbits can mate at the age of one month. Once they have started mating, the female gives birth to one male and one female each month. Assuming the rabbits never die during the year, how many pairs of rabbits will there be in one year?



At the end of the first month, there is still only one pair: they have matured and are now ready to mate. At the end of the second month, they have produced a new pair and there are 2 pairs of rabbits. At the end of the third month, the original pair have produced another pair, and the second pair are ready to mate, so there are 3 pairs. At the end of the fourth month, two pairs have produced a new pair, plus the third pair is ready to mate, so there are 5 pairs of rabbits.

Do you notice a pattern with the number of pairs of rabbits?

Let's write the pairs as a sequence:{1,1,2,3,5,...}

How can we find t_n recursively?

Notice that we obtain each term of the sequence by adding the two previous terms:

We begin with 1. (t_1)

$t_0 + t_1 = 0 + 1 = 1 = t_2$

$t_1 + t_2 = 1 + 1 = 2 = t_3$

$t_2 + t_3 = 1 + 2 = 3 = t_4$

$t_3 + t_4 = 2 + 3 = 5 = t_5$

In general, the rule is as follows:

$\mathbf{t}_n = \mathbf{t}_{n-2} + \mathbf{t}_{n-1}$

This sequence is known as the **Fibonacci Sequence**. Leonardo Bonacci, known as Fibonacci (which means “son of Bonacci”) was an Italian mathematician who first considered the rabbit example in the year 1202.

So, to find out how many pairs of rabbits there would be in a year, we need to find out t_{12} (since there are 12 months in a year).

$\{1,1,2,3,5,8,13,21,34,55,89,144\}$ so $t_{12} = 144$.

Problems

1. Given the sequence, write a rule in words:

a) {0,3,6,9,...}

b) {3,6,12,24,...}

c) {11,9,7,5,...}

2. Given the rule, write the first 4 terms of the sequence:

a) Starting at 5, add 6 to each term.

b) Each term is triple the term number.

c) The sequence of odd numbers without a 1 in them

3. Dora is practising her math. She wants to be able to do 26 questions in one day. On day 1, she does 2 questions. On day 2, she does 5 questions. On day 3, she does 8 questions. How many days will it take Dora to do 25 questions?

4. Describe the sequence with a closed-form definition. Then find the 8th term.

a) {0.5,1,1.5,2,..}

b) {5,8,11,14,...}

c) {8,9,10,11,...} d) {2,5,10,17,...}

5. Describe the sequence with a recursive definition. Then find the next term.:

a) {7,18,29,40,...}

b) {100,52,28,16,...}

c) {1,2,5,...}

6. Determine whether the following sequences are arithmetic or geometric:

a) {1,22,43,64,...}

b) {1,5,25,...}

c) {10,5,0,...}

7. Find the common difference and the nth term:

a) {75,69,63,57,...}

b) {7,19,31,..}

c) {4,4,4,4,...}

d) {0.1,0.2,0.3,0.4,...}

8. Find the common ratio and the next term:

a) {2,6,18,...}

b) {64,16,4,...}

c) {1,$\sqrt{2}$, 2,...}

9. If we only want the positive numbers in the sequence {95,84,73,...}, how many terms are in the sequence?

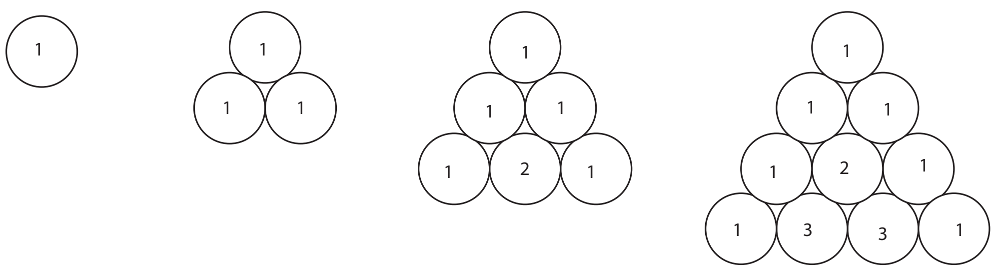
10. Given the sequence $t_n = 7 + 8(n\text{-}1)$:
a) what is the 11th term?
b) If $t_n = 63$, solve for n.

11. What is the sum of the sequence {1,2,3,4,5}?

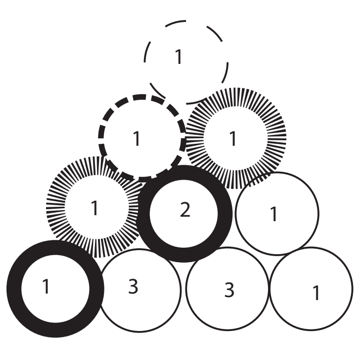
12. If we have the closed-form definition of a sequence $t_n = (n \times n)\text{-}5n+2$, what is the smallest value of n that makes t_n positive?

13. Annie is training for a 10-kilometre race. She has a weekly training program where she runs 2 kilometres on Monday, 4 kilometres on Tuesday, 6 kilometres on Wednesday, 8 kilometres on Thursday and 10 kilometres on Saturday. She continues this training sequence every week for 10 weeks. How far did she run on her 37th run?

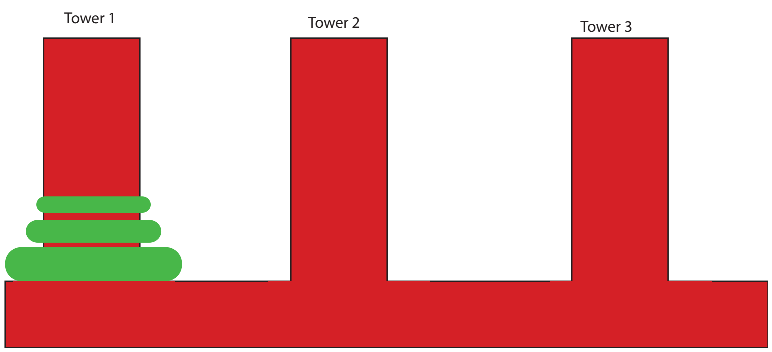
14. Below we have a sequence known as Pascal's Triangle:



a) Find the next 3 terms in Pascal's triangle
b) What is the rule of the sequence for the sum of each horizontal row?
c) Using the Pascal's Triangle below, find the sum of the terms with each special pattern, and create a sequence in ascending order (from smallest to largest). The circles with no pattern yet are not part of the sequence yet. That is, the first term is 1 (the dotted circle at the top), the second term is also 1 (the dashed-lined circle below it to the left).
(Hint: You should recognize the sequence)



15. The Tower of Hanoi is a game with three towers, and a certain amount of rings. The object of the game is to move all of the rings from the first tower to the third tower in the same order (largest at the bottom to smallest at the top). The catch: You can only move one peg at a time, and a ring can never have a larger ring on top of it.



The minimum number of moves for 1 ring is 1, for 2 rings is 3, and for 3 rings is 7. Find the closed-form definition of the sequence of minimum moves. (Hint: try working with exponents)

Solutions to Problems

1. a) Starting with 0, add 3 to each term to get the next term.
b) Starting with 3, multiply each term by 2 to get the next term.
c) Starting with 11, subtract 2 from each term to get the next term.

2. a) {5,11,17,23}
b) {3,6,9,12}
c){3,5,7,9}

3. {2,5,8,11,14,17,20,23,26}
So Dora will do 26 questions on day 9.

4. a) $t_n = n \div 2$
$t_8 = 4$
b) $t_n = 3n + 2$
$t_8 = 26$
c) $t_n = n + 7$
$t_8 = 15$
d) $t_n = (n \times n) + 1$
$t_8 = 65$

5. a) $t_n = t_{n-1} + 11$

$t_5 = 51$

b) $t_n = (t_{n-1} \div 2) + 2$

$t_5 = 10$

c) $t_n = (t_{n-1} \times t_{n-1}) + 1$

$t_4 = 26$

6. a) Arithmetic

b) Geometric

c) Arithmetic

7. a) d = -6

t_n = 75 - 6(n-1)

b) d = 12

t_n = 7 + 12(n-1)

c) d = 0

t_n = 4

d) d = 0.1

t_n = 0.1 + 0.1(n-1)

8. a) r = 3

$t_4 = 54$

b) $r = \frac{1}{4}$

$t_4 = 1$

c) $r = \sqrt{2}$

$t_4 = 2\sqrt{2}$

9. {95,84,73,62,51,40,29,18,7}

There are 9 terms in the sequence.

10. a) $t_{11} = 87$

b) 63 = 7 + 8(n-1)

63 - 7 = 8(n-1)

56 = 8(n-1)

$\frac{56}{8} = (n-1)$

7 = n - 1

n = 8

11. 1 + 2 + 3 + 4 + 5 = 15

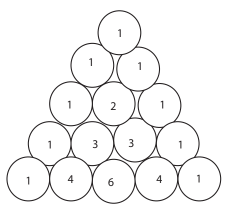
12. n = 5 :

(5 $\times$ 5) - (5 $\times$ 5) + 2 = 2, but

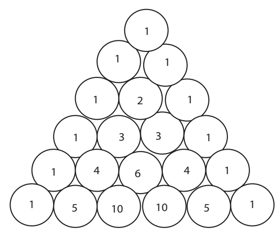
(4 $\times$ 4) - (5 $\times$ 4) + 2 = -2

13. On all days that are multiples of 5, she runs 10 kilometres. So, on the 35th day, she runs 10 kilometres. That means that on the 36th day, she runs 2 kilometres, and on the 37th day, she runs 4 kilometres.

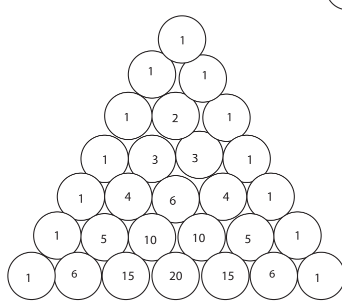
14. a)



t_5



t_6



t_7

b) {1,2,4,8,...}

$t_n = 2 \times t_{n-1}$

c){1,1,2,3,5,...} (Fibonacci Sequence)

15. $t_n = 2^n - 1$