Week 2 Problems - Robert Garbary, Fall 2014

• 1: Properties 1-3 of the norm in lecture notes, done during the talk. Go through the proofs I wrote down if you were confused when I explained them!

• 2: Show that Euclid’s lemma is false if we drop the assumption that $p$ is irreducible. More specifically, exhibit integers $a, b, c$ where $a|bc$ but neither $a|b$ nor $a|c$.

• 3: Is $2$ an irreducible element of $\mathbb{Z}[i]$? Is $3$ irreducible?

• 4: (A bit harder) Let $\mathbb{Z}[\sqrt{-5}]$ denote the collection of all ‘numbers’ of the form

$$a + b\sqrt{-5}$$

with $a, b \in \mathbb{Z}$ (a,b integers). We can again add and multiply two of these numbers, and get another number of this form. For example the addition looks like

$$1 + \sqrt{-5} + (2 - 4\sqrt{-5}) = (1 + 2) + (1 - 4)\sqrt{-5} = 3 - 3\sqrt{-5}$$

while the multiplication looks like

$$(1 + \sqrt{-5})(2 + 3\sqrt{-5}) = 1 \cdot 2 + 1 \cdot 3\sqrt{-5} + \sqrt{-5} \cdot 2 + \sqrt{-5} \cdot 3\sqrt{-5} = 3 + 3\sqrt{-5} + 2\sqrt{-5} + 3(-5) = (3 - 15) + (3 + 2)\sqrt{-5} = -12 + 5\sqrt{-5}$$

Prove that the only units of $\mathbb{Z}[\sqrt{-5}]$ are $1$ and $-1$. That is, prove the following: let $x, y \in \mathbb{Z}[\sqrt{-5}]$ satisfy $xy = 1$. Prove that $x$ and $y$ are both $\pm 1$.

• 5: Show that $\mathbb{Z}[\sqrt{-5}]$ does not have the unique factorization property by finding two different factorizations of $6$. 