1. Prove by induction that the sum of the first \( n \) perfect squares is \( \frac{n(n+1)(2n+1)}{6} \). What form of induction did you use?

**Solution:**

The base case is where \( n = 0 \), which clearly holds since \( 0 = 0^2 \). If you don’t like that, we could use \( n = 1 \), which has \( \frac{1 \cdot 2 \cdot 3}{6} = 1 \), which is the sum of the first square.

Assume (using weak induction) that the result holds for some \( n = k \) (\( k \leq 1 \)).

We must prove the result for \( n = k + 1 \). Notice that:

\[
\begin{align*}
1^2 + 2^2 + \cdots + (k+1)^2 &= 1^2 + 2^2 + \cdots k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{6(k+1)^2 + k(k+1)(2k+1)}{6} \\
&= \frac{(k+1)[6(k+1) + k(2k+1)]}{6} \\
&= \frac{(k+1)[6k + 6 + 2k^2 + k]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k+1)[(k + 2)(2k + 3)]}{6} \\
&= \frac{(k + 1)[(k + 2)(2k + 1) + 1]}{6}
\end{align*}
\]

which proves the result.

2. Find and prove a closed form for \( a_n = 2\sqrt{a_{n-1}} \) with \( a_0 = 2 \).

**Solution:** Let’s start by making a table:

\[
\begin{align*}
a_0 &= 2 \\
a_1 &= 2\sqrt{a_0} = 2^{3/2} \\
a_2 &= 2\sqrt{a_1} = 2^{7/4} \\
a_3 &= 2\sqrt{a_2} = 2^{15/8} \\
a_4 &= 2\sqrt{a_3} = 2^{31/16}
\end{align*}
\]

Thus, we guess that \( a_n = 2^{(2(n+1)-1)/2^n} = 2^{2-1/2^n} \).
We can prove this by induction. When \( n = 0 \), we have \( a_0 = 2 = 2^{2-1/2^0} = 2^{2-1} = 2^1 \).
Assume the result holds for \( n = k \) (\( k \geq 0 \)).
We prove the result holds for \( n = k + 1 \).
We have:

\[
a_{k+1} = 2\sqrt{a_k} \\
= 2 \cdot \sqrt{2^{2-1/2^k}} \\
= 2 \cdot 2^{(2-1/2^k)/2} \\
= 2 \cdot 2^{1-1/2^{k+1}} \\
= 2^{2-1/2^{k+1}}
\]

which proves the result.

3. (a) Solve the recurrence \( a_n = a_{n-1} + 2a_{n-2} \) with \( a_0 = 2 \) and \( a_1 = 7 \).

**Solution:** Since it is a linear homogeneous recurrence, we find the characteristic equation of:

\[ r^2 - r - 2 = 0 \]

which factors as \((r + 1)(r - 2) = 0\), and we have roots \( r_1 = 2 \) and \( r_2 = -1 \).
From the theorem discussed in lecture, we know that \( a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n \) is a solution. We know

\[
a_0 = \alpha_1 + \alpha_2 = 2 \\
a_1 = 2\alpha_1 - \alpha_2 = 7
\]

Adding together these equations, we see that \(3\alpha_1 = 9\) and so \( \alpha_1 = 3\). Thus, \( \alpha_2 = -1\). Thus, the closed form for this recurrence is:

\[ a_n = 3 \cdot 2^n - (-1)^n. \]

(b) Redo part (a), except this time, change \( a_0 = 10 \) and \( a_1 = 4 \).

**Solution:** Everything involving the characteristic equation and roots is the same as before, except we have a different set of equations to solve in order to determine the coefficients.
That is, we have to now solve:

\[
a_0 = \alpha_1 + \alpha_2 = 10 \\
a_1 = 2\alpha_1 - \alpha_2 = 4
\]

Again, adding together these two equations, we have \(3\alpha_1 = 14\) and thus \( \alpha_1 = \frac{14}{3} \) and \( \alpha_2 = \frac{16}{3}. \)
Thus, the closed form for this recurrence is:

\[ a_n = \frac{14}{3} \cdot 2^n + \frac{16}{3} (-1)^n. \]
(c) Redo part (a), except this time, change \( a_0 = a \) and \( a_1 = b \).

So, in this general case, we have:

\[
\begin{align*}
a_0 &= \alpha_1 + \alpha_2 = a \\
a_1 &= 2\alpha_1 - \alpha_2 = b
\end{align*}
\]

Again, adding together these two equations, we have \( 3\alpha_1 = a + b \), and so \( \alpha_1 = \frac{a + b}{3} \) and then \( \alpha_2 = \frac{2a - b}{3} \).

Thus, the closed form for this recurrence is:

\[
a_n = \frac{a + b}{3} \cdot 2^n + \frac{2a - b}{3} (-1)^n.
\]

4. Solve the recurrence \( a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3} \) with \( a_0 = 8 \), \( a_1 = 6 \) and \( a_2 = 26 \).

**Solution:**

Since it is a linear homogeneous recurrence, we find the characteristic equation of:

\[
r^3 + r^2 - 4r - 4 = 0
\]

which we can rewrite as:

\[
r^3 + -4r + r^2 - 4 = r(r^2 - 4) + (r^2 - 4) = 0
\]

which factors as \( (r + 1)(r - 2)(r + 2) \).

From our theorem, we know that

\[
a_n = \alpha_1 (-1)^n + \alpha_2 (-2)^n + \alpha_3 (2^n)
\]

is a solution. We need to use the initial conditions to solve this.

\[
\begin{align*}
a_0 &= \alpha_1 + \alpha_2 + \alpha_3 = 8 \\
a_1 &= -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6 \\
a_2 &= \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26
\end{align*}
\]

After multiplying the first equation by 4 and then subtracting the third equation, we get \( \alpha_1 = 2 \). Substituting and combine the second and third equations, we get \( \alpha_2 = 1 \), and then \( \alpha_3 = 5 \).

Thus the closed form of the recurrence is:

\[
a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot (2^n).
\]

5. Solve the recurrence \( a_n = 6a_{n-1} - 9a_{n-2} \) with \( a_0 = 1 \) and \( a_1 = 6 \).

**Solution:**

We find the characteristic equation as:

\[
r^2 - 6r + 9 = 0
\]
Which factors as \((r - 3)^2 = 0\), and thus 3 is a solution with multiplicity 2.

By our theorem in lecture, we know that if we have root with multiplicity \(m\), the solution involves a polynomial of degree \(m - 1\). So, the closed form is:

\[
a_n = (\alpha_{10} + \alpha_{11}n)3^n
\]

From this, we use our initial conditions to get:

\[
a_0 = \alpha_{10} = 1
\]
\[
a_1 = 3\alpha_{10} + 3\alpha_{11} = 6
\]

Thus, \(\alpha_{10} = 1\) and \(\alpha_{11} = 1\).

Thus, the closed form is:

\[
a_n = 3^n + n3^n
\]

Try it out for yourself and verify it!

6. Solve the recurrence \(s_k = as_{k-1}\) for any value of \(a\). Using that solution, solve \(s_k = as_{k-1} + 1\).

*Errata: The question should have given some initial condition, like \(s_0 = 1\).*

**Solution:**

The characteristic equation is very simple here: \(r - a = 0\), which means that \(r = a\). Thus, the solution is of the form \(s_k = a\alpha^n\), and if \(s_0 = 1\), then \(a = 1\), and the recurrence is \(s_k = a^n\), which is not too surprising (i.e., the sequence grows exponentially in \(a\)).

For the variant, if \(s_k = as_{k-1} + 1\), then we notice that it should be growing exponentially. However, notice the first few terms:

\[
s_0 = 1
\]
\[
s_1 = a + 1
\]
\[
s_2 = a(a + 1) + 1 = a^2 + a + 1
\]
\[
s_3 = a(a^2 + a + 1) + 1 = a^3 + a^2 + a + 1
\]

Thus, we can notice (and fairly easily prove using induction) that \(s_k = \sum_{i=0}^{k} a^i\) is the closed form.

7. Solve \(a_n = 2a_{n-1} - a_{n-2} + 2^n\) with \(a_0 = 1\) and \(a_1 = 2\).

**Solution:**

Notice that this is a linear non-homogeneous recurrence, we need to find a recurrence \(b_n\) similar to \(2^n\). Let us guess:

\[
b_n = c2^n + d.
\]

Thus

\[
\begin{align*}
b_n &= 2b_{n-1} - b_{n-2} + 2^n \\
\Leftrightarrow c2^n + d &= 2(c2^{n-1} + d) - (c2^{n-2} + d) + 2^n \\
\Leftrightarrow c2^n + d &= c2^n + 2d - c2^{n-2} - d + 2^n \\
\Leftrightarrow 0 &= (-4c + 4c - c + 4)2^{n-2} + (-d + 2d - d)
\end{align*}
\]

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which must be true for all $n$. thus, $d = 0$ and $c = 4$. Therefore, $b_n = 4 \cdot 2^n$.

We really would like to find a closed form for $a_n$, not $b_n$. Notice that $a_n = b_n + h_n$, where $h_n$ is the homogeneous recurrence $h_n = 2h_{n-1} - h_{n-2}$.

The characteristic equation is:

$$r^2 - 2r + 1 = 0$$

$$\Rightarrow (r - 1)^2 = 0$$

Thus, $r = 1$ is the root with multiplicity 2. Thus $h_n = (\alpha_1 + \alpha_2)(1)^n = \alpha_1 + \alpha_2n$ is a solution. Thus

$$a_n = b_n + h_n = 4 \cdot 2^n + \alpha_1 + \alpha_2n$$

is a solution.

Using the initial conditions, we know:

$$a_0 = 4 + \alpha_1 = 1$$
$$a_1 = 8 - \alpha_1 + \alpha_2 = 2$$

Adding these two equations together, we get $\alpha_2 = -3$, which yields $\alpha_1 = -3$.

Thus, the closed form is:

$$a_n = 4 \cdot 2^n - 3n - 3.$$  

8. If $\varphi = \frac{1 + \sqrt{5}}{2}$, find $\frac{-1}{\varphi}$. Then, find $1 - \varphi$. Be amazed.

**Solution:**

$$\frac{1}{\varphi} = \frac{2}{1 + \sqrt{5}}$$

$$= \frac{2(1 - \sqrt{5})}{(1 + \sqrt{5})(1 - \sqrt{5})}$$

$$= \frac{2(1 - \sqrt{5})}{1 - 5}$$

$$= \frac{2(1 - \sqrt{5})}{4}$$

$$= \frac{(1 - \sqrt{5})}{2}$$

and

$$1 - \varphi = 1 - \frac{1 + \sqrt{5}}{2}$$

$$= \frac{2 - (1 + \sqrt{5})}{2}$$

$$= \frac{1 - \sqrt{5}}{2}$$
and the amazing thing is that they are equal!