Math Circles
29 October 2014
Recurrences, Part 1
Instructor: Troy Vasiga
SOLUTIONS

1. Prove by induction that the sum of the first n perfect squares is $\frac{n(n+1)(2n+1)}{6}$. What form of induction did you use?

   Solution:

   The base case is where $n = 0$, which clearly holds since $0 = 0^2$. If you don't like that, we could use $n = 1$, which has $\frac{1 \cdot 2 \cdot 3}{6} = 1$, which is the sum of the first square.

   Assume (using weak induction) that the result holds for some $n = k$ ($k \leq 1$).

   We must prove the result for $n = k + 1$. Notice that:

$$\begin{aligned}
1^2 + 2^2 + \cdots + (k+1)^2 &= 1^2 + 2^2 + \cdots k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{6(k+1)^2 + k(k+1)(2k+1)}{6} \\
&= \frac{(k+1)[6(k+1) + k(2k+1)]}{6} \\
&= \frac{(k+1)[6k + 6 + 2k^2 + k)]}{6} \\
&= \frac{(k+1)[2k^2 + 7k + 6)]}{6} \\
&= \frac{(k+1)[(k+2)(2k+3)]}{6} \\
&= \frac{(k+1)[(k+2)(2(k+1)+1]}{6}
\end{aligned}$$

   which proves the result.

2. Find and prove a closed form for $a_n = 2\sqrt{a_{n-1}}$ with $a_0 = 2$.

   Solution: Let's start by making a table:

$$\begin{aligned}
a_0 &= 2 \\
a_1 &= 2\sqrt{a_0} = 2^{3/2} \\
a_2 &= 2\sqrt{a_1} = 2^{7/4} \\
a_3 &= 2\sqrt{a_1} = 2^{15/8} \\
a_4 &= 2\sqrt{a_2} = 2^{31/16}
\end{aligned}$$

   Thus, we guess that $a_n = 2^{(2^(n+1)-1)/2^n} = 2^{2-1/2^n}$.

We can prove this by induction. When $n = 0$, we have $a_0 = 2 = 2^{2-1/2^0} = 2^{2-1} = 2^1$.

Assume the result holds for $n = k$ ($k \geq 0$).

We prove the result holds for $n = k + 1$.

We have:

$$\begin{aligned} a_{k+1} &= 2\sqrt{a_k} \\ &= 2 \cdot \sqrt{2^{2-1/2^k}} \\ &= 2 \cdot 2^{(2-1/2^k)/2} \\ &= 2 \cdot 2^{1-1/2^{k+1}} \\ &= 2^{2-1/2^{k+1}} \end{aligned}$$

which proves the result.

3. (a) Solve the recurrence $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Solution: Since it is a linear homogeneous recurrence, we find the characteristic equation of:

$$r^2 - r - 2 = 0$$

which factors as $(r + 1)(r - 2) = 0$, and we have roots $r_1 = 2$ and $r_2 = -1$.

From the theorem discussed in lecture, we know that $a_n = \alpha_1(2)^n + \alpha_2(-1)^n$ is a solution.

We know

$$\begin{aligned} a_0 &= \alpha_1 + \alpha_2 = 2 \\ a_1 &= 2\alpha_1 - \alpha_2 = 7 \end{aligned}$$

Adding together these equations, we see that $3\alpha_1 = 9$ and so $\alpha_1 = 3$. Thus, $\alpha_2 = -1$. Thus, the closed form for this recurrence is:

$$a_n = 3 \cdot 2^n - (-1)^n.$$

(b) Redo part (a), except this time, change $a_0 = 10$ and $a_1 = 4$.

Solution: Everything involving the characteristic equation and roots is the same as before, except we have a different set of equations to solve in order to determine the coefficients.

That is, we have to now solve:

$$\begin{aligned} a_0 &= \alpha_1 + \alpha_2 = 10 \\ a_1 &= 2\alpha_1 - \alpha_2 = 4 \end{aligned}$$

Again, adding together these two equations, we have $3\alpha_1 = 14$ and thus $\alpha_1 = \frac{14}{3}$ and $\alpha_2 = \frac{16}{3}$.

Thus, the closed form for this recurrence is:

$$a_n = \frac{14}{3} \cdot 2^n + \frac{16}{3}(-1)^n.$$

(c) Redo part (a), except this time, change $a_0 = a$ and $a_1 = b$.
So, in this general case, we have:

$$\begin{aligned} a_0 &= \alpha_1 + \alpha_2 = a \\ a_1 &= 2\alpha_1 - \alpha_2 = b \end{aligned}$$

Again, adding together these two equations, we have $3\alpha_1 = a + b$, and so $\alpha_1 = \frac{a+b}{3}$ and then $\alpha_2 = \frac{2a-b}{3}$.
Thus, the closed form for this recurrence is:

$$a_n = \frac{a+b}{3} \cdot 2^n + \frac{2a-b}{3}(-1)^n.$$

4. Solve the recurrence $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ with $a_0 = 8$, $a_1 = 6$ and $a_2 = 26$.

Solution:

Since it is a linear homogeneous recurrence, we find the characteristic equation of:

$$r^3 + r^2 - 4r - 4 = 0$$

which we can rewrite as:

$$r^3 + -4r + r^2 - 4 = r(r^2 - 4) + (r^2 - 4) = 0$$

which factors as $(r+1)(r-2)(r+2)$.

From our theorem, we know that

$$a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha^3(2^n)$$

is a solution. We need to use the initial conditions to solve this.

$$\begin{aligned} a_0 &= \alpha_1 + \alpha_2 + \alpha_3 = 8 \\ a_1 &= -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6 \\ a_2 &= \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26 \end{aligned}$$

After multiplying the first equation by 4 and then subtracting the third equation, we get $\alpha_1 = 2$. Substituting and combine the second and third equations, we get $\alpha_2 = 1$, and then $\alpha_3 = 5$.

Thus the closed form of the recurrence is:

$$a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot (2^n).$$

5. Solve the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$.

Solution:

We find the characteristic equation as:

$$r^2 - 6r + 9 = 0$$

Which factors as $(r-3)^2 = 0$, and thus 3 is a solution with multiplicity 2.

By our theorem in lecture, we know that if we have root with multiplicity m, the solution involves a polynomial of degree $m-1$. So, the closed form is:

$$a_n = (\alpha_{10} + \alpha_{11} n)(3)^n$$

From this, we use our initial conditions to get:

$$\begin{aligned} a_0 &= \alpha_{10} = 1 \\ a_1 &= 3\alpha_{10} + 3\alpha_{11} = 6 \end{aligned}$$

Thus, $\alpha_{10} = 1$ and $\alpha_{11} = 1$.

Thus, the closed form is:

$$a_n = 3^n + n3^n$$

Try it out for yourself and verify it!

6. Solve the recurrence $s_k = as_{k-1}$ for any value of a. Using that solution, solve $s_k = as_{k-1} + 1$. *Errata: The question should have given some initial condition, like $s_0 = 1$.*

   Solution:

   The characteristic equation is very simple here: $r - a = 0$, which means that $r = a$. Thus, the solution is of the form $s_k = \alpha a^n$, and if $s_0 = 1$, then $\alpha = 1$, and the recurrence is $s_k = a^n$, which is not too surprising (i.e., the sequence grows exponentially in a).

   For the variant, if $s_k = as_{k-1} + 1$, then we notice that it should be growing exponentially. However, notice the first few terms:

$$\begin{aligned} s_0 &= 1 \\ s_1 &= a + 1 \\ s_2 &= a(a+1) + 1 = a^2 + a + 1 \\ s_3 &= a(a^2 + a + 1) + 1 = a^3 + a^2 + a + 1 \end{aligned}$$

   Thus, we can notice (and fairly easily prove using induction) that $s_k = \sum_{i=0}^{k} a^i$ is the closed form.

7. Solve $a_n = 2a_{n-1} - a_{n-2} + 2^n$ with $a_0 = 1$ and $a_1 = 2$.

   Solution:

   Notice that this is a linear non-homogeneous recurrence, we need to find a recurrence b_n similar to 2^n. Let us guess:

$$b_n = c2^n + d.$$

   Thus

$$\begin{aligned} b_n &= 2b_{n-1} - b_{n-2} + 2^n \\ \Leftrightarrow c2^n + d &= 2(c2^{n-1} + d) - (c2^{n-2} + d) + 2^n \\ \Leftrightarrow c2^n + d &= c2^n + 2d - c2^{n-2} - d + 2^n \\ \Leftrightarrow 0 &= (-4c + 4c - c + 4)2^{n-2} + (-d + 2d - d) \end{aligned}$$

which must be true for all n. thus, $d = 0$ and $c = 4$. Therefore, $b_n = 4 \cdot 2^n$.

We really would like to find a closed form for a_n, not b_n. Notice that $a_n = b_n + h_n$, where h_n is the homogeneous recurrence $h_n = 2h_{n-1} - h_{n-2}$.

The characteristic equation is:

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ (r-1)^2 &= 0 \end{aligned}$$

Thus, $r = 1$ is the root with multiplicity 2. Thus $h_n = (\alpha_1 + \alpha_2)(1)^n = \alpha_1 + \alpha_n$ is a solution.

Thus

$$a_n = b_n + h_n = 4 \cdot 2^n + \alpha_1 + \alpha_2 n$$

is a solution.

Using the initial conditions, we know:

$$\begin{aligned} a_0 &= 4 + \alpha_1 = 1 \\ a_1 &= 8 - \alpha_1 + \alpha_2 = 2 \end{aligned}$$

Adding these two equations together, we get $\alpha_2 = -3$, which yields $\alpha_1 = -3$.

Thus, the closed form is:

$$a_n = 4 \cdot 2^n - 3n - 3.$$

8. If $\varphi = \frac{1+\sqrt{5}}{2}$, find $-\frac{1}{\varphi}$. Then, find $1 - \varphi$. Be amazed.

Solution:

$$\begin{aligned} -\frac{1}{\varphi} &= -\frac{2}{1+\sqrt{5}} \\ &= -\frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} \\ &= -\frac{2(1-\sqrt{5})}{(1-5)} \\ &= \frac{2(1-\sqrt{5})}{4} \\ &= \frac{(1-\sqrt{5})}{2} \end{aligned}$$

and

$$\begin{aligned} 1 - \varphi &= 1 - \frac{1+\sqrt{5}}{2} \\ &= \frac{2-(1+\sqrt{5})}{2} \\ &= \frac{1-\sqrt{5}}{2} \end{aligned}$$

and the amazing thing is that they are equal!