Grade 11/12 Math Circles

Fall 2014 - Nov. 12

Recurrences, Part 3

Definition of an L-system

An *L-system* or *Lindenmayer system* is a parallel rewriting system.

By parallel, we mean we each step involves replacing every possible occurrence at the same time.

We also need to specify the starting condition/symbol.

Rewriting Rules

We use the rule

\[ X \rightarrow Y \]

to mean “replace every occurrence of \( X \) with \( Y \).”

For example, if we have the rule \( X \rightarrow ABX \) and we have the current word:

\[ ABXBAX \]

then the next word will be

\[ ABABXBAABX \]
What does this sequence do?

1. \( A \rightarrow AB \)

2. \( B \rightarrow \epsilon \)

(Here \( \epsilon \) means “this disappears.”)

Start with \( A \rightarrow AB \rightarrow AB \rightarrow AB \)

Not much.

This just generates \( AB \) at the first step, which becomes \( AB \) after the second step, and so on.

So, this is just the word \( AB \).

This indicates that \( AB \) is a fixed point of this recurrence.

Definition: A fixed point of a function \( f \) is a value \( t \) such that \( f(t) = t \).

Notice: if \( f(t) = t \), then \( f(f(t)) = f(t) = t \), which means \( f^n(t) = f(f(\cdots f(t))\cdots) = t \).

What does this sequence do?

1. \( A \rightarrow AB \)

2. \( B \rightarrow B \)

Start with \( A \rightarrow AB \rightarrow ABB \rightarrow ABBB \rightarrow \ldots \)

After \( n \) iterations of the rewriting rules, we have the word

\[ AB^n = ABB \cdots B. \]
What does this sequence do?

1. $A \rightarrow B$

2. $B \rightarrow AB$

Start with $A \rightarrow B \rightarrow AB \rightarrow BAB \rightarrow ABBAB \rightarrow BABABBAB$

Notice what happens when we find the sum of the number of occurrences of $A$ and $B$.

<table>
<thead>
<tr>
<th>Sequence</th>
<th>A</th>
<th>B</th>
<th>A + B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>AB</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>BAB</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>ABBAB</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>BABABBAB</td>
<td>3</td>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

This forms a Fibonacci Sequence. Call the sum of the number of occurrences of $A$ and $B$ $f_n$ for each term of the sequence.

Also, notice the ratio

$$\frac{|A|}{|B|} = \frac{f_{n-1}}{f_{n-2}}$$

Where $|A|$ and $|B|$ are the number of occurrences of $A$ and $B$, respectively, in each term of the sequence.

$$\lim_{n \to \infty} \frac{f_{n-1}}{f_{n-2}} = \varphi$$
What does this sequence do?

1. \( A \rightarrow ABA \)

2. \( B \rightarrow BBB \)

Start with \( A \rightarrow ABA \rightarrow ABABBBABA \rightarrow ABABBBABABBBBBBBBBABABBBABA \)

This sequence forms what’s called Cantor Dust.

**Cantor Dust**

The line at the top of the figure represents the first term in the sequence, \( A \). The line below represents the second term in the sequence, \( ABA \), where \( B \) represents a break in the line above it. This then recurses.

How much of the line is removed?

\[
\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}
\]

But this is just a geometric series with the first term being \( a = \frac{1}{3} \) and the common ratio being \( r = \frac{2}{3} \). This series is equal to

\[
\frac{a}{1 - r} = \frac{\frac{1}{3}}{1 - \frac{2}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1 = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}
\]

Therefore, this whole line will be removed.

Will a length of \( \frac{1}{4} \) ever be removed for instance?

It will never be removed since \( \frac{2}{9} < \frac{1}{4} < \frac{1}{3} \).
What does this sequence do?

1. $X \rightarrow X + YF$

2. $Y \rightarrow FX - Y$

Start with $FX \rightarrow FX + YF \rightarrow FX + YF + FX - YF$
$\rightarrow FX + YF + FX - YF + FX + YF - FX - YF$.

Then we take away the $X$s and $Y$s to get

$F + F + F - F + F + F - F - F$

This sequence will form what’s called a dragon curve.

**Dragon Curve**

Do the following things:

- $F$ means move forward
- $+$ means turn clockwise $90^\circ$
- $-$ means turn counterclockwise $90^\circ$

Here are some steps:
This is what the figure will look like after we perform many steps.

**Another L-system**

Start with:

\[ F -- F -- F \]

One rule:

\[ F \rightarrow F + F -- F + F \]

Let:

- \( F \): move forward
- \( + \): move clockwise by \( \frac{\pi}{3} \)
- \( - \): move counterclockwise by \( \frac{\pi}{3} \)

This forms what’s called a **Koch Snowflake**:
What does this sequence do?

1. $0 \rightarrow 01$

2. $1 \rightarrow 10$

Start $0 \rightarrow 01 \rightarrow 0110 \rightarrow 01101001 \rightarrow 0110100110010110$.

This is called the Prouhet-Thue-Morse Sequence and it does lot of things.

**Prouhet-Thue-Morse Sequence**

Call the sequence $t = t_0t_1t_2... = 0 1 1 0 1 0 0 1 ...$

**Draw a picture**

- If $t_n = 0$, turn by $\pi$
- If $t_n = 1$, move ahead one unit and then rotate counterclockwise by an angle of $\frac{\pi}{3}$.

This forms what’s called a **Koch Curve**
**Another definition of the PTM Sequence**

Write out all integers in base-2 (i.e., binary).

\[
0, 1, 10, 11, 100, 101, 110, 111, 1000, 1001, \ldots
\]

Define \( s_2(x) \) to the sum of the digits in the base-2 representation of \( x \).

\[
0 \ 1 \ 1 \ 0 \ 1 \ 2 \ 2 \ 3 \ 1 \ 2 \ldots
\]

If we did this for integers [0,16] keeping in mind that \( 2, 4 = 0 \) and \( 3 = 1 \) in base-2 then we get.

\[
0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0
\]

**Yet another definition of the PTM Sequence**

Write the number, then flip the bits and rewrite it.

Define:

\[
\begin{align*}
X_0 &= 0 \\
X_{n+1} &= X_n \bar{X}_n
\end{align*}
\]

where \( \bar{x} \) means change all 0's to 1's, and 1's to 0's.

\[
0 \rightarrow 01 \rightarrow 0110 \rightarrow 01101001 \rightarrow 0110100110010110
\]
Equivalent Definitions

Prove that the last two definitions are equivalent. In other words, prove:

\[ t_n = s_2(n) \mod 2 \]

where \( t_n \) is the \( n \)th term of \( X \) as defined on the previous slide.

**Proof:** Use strong mathematical induction.

Base cases:
If \( n = 0 \), \( s_2(0) = 0 = t_0 \).
If \( n = 1 \), \( s_2(1) = 1 = t_1 \).

Assume true for all \( n < 2^k \), \( k \geq 1 \). That is \( s_2(n) = t_n \) for all \( n < 2^k \).

We will show the result holds for any \( n \) such that \( 2^k \leq n < 2^{k+1} \).

Now, \( t_n \) is the \( n \)th symbol of \( X_{k+1} = X_k \bar{X}_k \).
Thus, \( t_n \) is the \((n - 2^k)\)th symbol of \( \bar{X}_k \), which means that \( t_n = (t_{n-2^k} + 1) \mod 2 \).

By induction, we know that \( t_{n-2^k} = s_2(n - 2^k) \mod 2 \).

Since \( 2^k \leq n < 2^{k+1} \), we have \( s_2(n) = s_2(n - 2^k) + 1 \), since the subtraction of \( 2^k \) will remove the leading 1 in the base-2 representation, but keep all other digits the same.

Thus,

\[
\begin{align*}
t_n &= (t_{n-2^k} + 1) \mod 2 \\
&= (s_2(n - 2^k) + 1) \mod 2 \\
&= ((s_2(n) + 1) + 1) \mod 2 \\
&= s_2(n) \mod 2
\end{align*}
\]

which proves the result.
Morphism

A morphism is a map $h$ on strings that satisfies the identity

$$h(xy) = h(x)h(y)$$

for all strings $x, y$.

**PTM morphism**

Define the PTM Morphism

$$\mu(0) = 01, \quad \mu(1) = 10.$$ 

Then

$$\mu(0) = 01$$
$$\mu^2(0) = \mu(\mu(0)) = 0110$$
$$\mu^3(0) = 010100$$
$$\mu^4(0) = 010100110010110$$

**Fixed Point of a Morphism**

The fixed point of a morphism $h$ beginning with $a$ is:

$$h^{\omega}(a) = \lim_{m \to \infty} h^m(a).$$
Equivalent Definitions

Let’s prove $\mu^n(0) = X_n$.

Use mathematical induction.

Base case: $n = 0$. $\mu^0(0) = 0 = X_0$.

Assume the result holds for $n = k$. That is, $\mu^k(0) = X_k$.

Prove true for $n = k + 1$.

\[
\begin{align*}
\mu^{k+1}(0) &= \mu^k(\mu(0)) \\
&= \mu^k(01) \\
&= \mu^k(0)\mu^k(1) \\
&= X_k \bar{X}_k \\
&= X_{k+1}
\end{align*}
\]

Overlap-Free

Let $x$ be any word

A square is a word of the form $xx$.

For example:

murmur, mama, papa, dada, hotshots

A cube is a word of the form $xxx$.

For example:

lalala
An overlap is a word $axaxa$, with a single letter $a$ and $x$ being any word.

For example, if $x = ma$ and $a = a$, then

“mamam” is an overlap

**How does this relate to the Thue-Morse sequence?**

The Thue-Morse sequence is overlap-free.

**Last Problem**

Consider the sequence:

$$
\begin{array}{cccc}
1 & 1/2 & 1/2 & 1/2 \\
2 & 3/4 & 3/4 & 3/4 \\
3/4 & 5/6 & 7/8 & 7/8 \\
5/6 & 7/8 & 9/10 & 9/10 \\
7/8 & 9/10 & 11/12 & 11/12 \\
9/10 & 11/12 & 13/14 & 13/14 \\
11/12 & 13/14 & 15/16 & 15/16
\end{array}
$$

What does this converge to?

First, observe the limit is:

$$\Pi_{n\geq0} \left( \frac{2n+1}{2n+2} \right)^{(-1)^{r_n}}$$

Let:

$$P = \Pi_{n\geq0} \left( \frac{2n+1}{2n+2} \right)^{(-1)^{r_n}}$$

$$Q = \Pi_{n\geq0} \left( \frac{2n}{2n+1} \right)^{(-1)^{r_n}}$$

Notice that

$$PQ = \frac{1}{2} \Pi_{n\geq1} \left( \frac{n}{n+1} \right)^{(-1)^{r_n}}$$
Rewrite, breaking over odd and even indices:

\[
PQ = \frac{1}{2} \prod_{n \geq 0} \left(\frac{2n + 1}{2n + 2}\right)^{(-1)^{2n+1}} \prod_{n \geq 1} \left(\frac{2n}{2n + 1}\right)^{(-1)^n}
\]

\[
= \frac{1}{2} P^{-1} Q.
\]

Thus, \( P^2 = \frac{1}{2} \).