Recurrences, Part 3

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Outline

• Selected solutions to problems from last week
• L-systems
• Examples of L-systems
• Prouhet-Thue-Morse sequence
• Other definitions of PTM
• Morphisms
• Squares, cubes and overlaps
• Summary
Use the Master Theorem to find the asymptotic running time for the recurrence

\[ T(n) = 4T(n/2) + n. \]

**Solution:**
We have \( a = 4 \), \( b = 2 \) and \( f(n) = n \).
Solution to Problem 5

Use the Master Theorem to find the asymptotic running time for the recurrence

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**Solution:**

We have $a = 4$, $b = 2$ and $f(n) = n$.

We calculate $n^{\log_b a} = n^{\log_2 4} = n^2$. 

Thus, we are in case 1 of the Master Theorem, and thus $T(n) = \Theta(n^2)$. 

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Then, \( f(n) = n \in O(n^{2-\epsilon}) \) for \( \epsilon = 0.9 \) (for instance).
Thus, we are in case 1 of the Master Theorem, and thus \( T(n) = \Theta(n^{\log_b a}) = \Theta(n^2) \).
Solution to Problem 7

Use the Master Theorem to find the asymptotic running time for the recurrence

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Then, \( f(n) = n^3 \in \Omega(n^{2+\epsilon}) \) for \( \epsilon = 1 \), for instance.
Solution to Problem 7

Use the Master Theorem to find the asymptotic running time for the recurrence

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**Solution:**

We have \( a = 4 \), \( b = 2 \) and \( f(n) = n^3 \).

We calculate \( n^{\log_b a} = n^{\log_2 4} = n^2 \).

Then, \( f(n) = n^3 \in \Omega(n^{3-\epsilon}) \) for \( \epsilon = 1 \), for instance.

Thus, we are in case 3 of the Master Theorem. We just need to verify that the second condition holds: that is, we need to show that

\[ af \left( \frac{n}{b} \right) \leq cf(n) \]

for some fixed \( c \) for all \( n \) sufficiently large.
Solution to Problem 7 (continued)

Plugging in our constants and \( f(n) = n^3 \), we have to show:

\[
4 \left( \frac{n}{2} \right)^3 \leq cn^3.
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Solution to Problem 7 (continued)

Plugging in our constants and \( f(n) = n^3 \), we have to show:

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Pick \( c = 1 \) and then for \( n > 1 \), we have:

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\frac{n^3}{2} \leq cn^3.
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Plugging in our constants and $f(n) = n^3$, we have to show:

$$4 \left( \frac{n}{2} \right)^3 \leq cn^3.$$ 

Pick $c = 1$ and then for $n > 1$, we have:

$$\frac{n^3}{2} \leq cn^3.$$ 

Thus, by case 3 of the Master Theorem, $T(n) = \Theta(n^3)$. 
Definition of an L-system

An L-system or Lindenmayer system is a parallel rewriting system.

By parallel, we mean we each step involves replacing every possible occurrence at the same time.

We also need to specify the starting condition/symbol.
Rewriting Rules

We use the rule

\[ X \rightarrow Y \]

to mean “replace every occurrence of \( X \) with \( Y \).”

For example, if we have the rule \( X \rightarrow ABX \) and we have the current word:

\[ ABXBAX \]

then the next word will be
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then the next word will be

$$ABABXBAABX$$
What does this sequence do?

1. $A \rightarrow AB$
2. $B \rightarrow \epsilon$

(Here $\epsilon$ means “this disappears.”)

Start with $A \Rightarrow AB \Rightarrow AB \Rightarrow AB$
Not much

This just generates $AB$ at the first step, which becomes $AB$ after the second step, and so on.

So, this is just the word $AB$. 
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Definition: A *fixed point* of a function $f$ is a value $t$ such that $f(t) = t$.

Notice: if $f(t) = t$, then $f(f(t)) = f(t) = t$, which means $f^n(t) = f(f(\cdots f(t))\cdots) = t$. 

What does this sequence do?

1. $A \rightarrow AB$

2. $B \rightarrow B$

Start with $A$
Infinitely more
Infinitely more

After $n$ iterations of the rewriting rules, we have the word

$$AB^n = ABB \cdots B.$$
What does this sequence do?

1. $A \rightarrow B$
2. $B \rightarrow AB$

Start with $A$.

Hint: write out the first few terms and look at them.
Fibonacci

Notice the length.

Notice the ratio

\[ \frac{|A|}{|B|} = \frac{f_{n-1}}{f_{n-2}} \leq \lim_{n \to \infty} = \varphi \]
What does this sequence do?

1. \( A \rightarrow ABA \)
2. \( B \rightarrow BBB \)

Start with \( A \).

\[ \Rightarrow \ ABA \Rightarrow \ ABA \ BBA \ \ ABA \]

\[ \Rightarrow \ ABA \ BBA \ ABA \ \ BBBBBBBBABA \ ABA \ BBA \]

\[ \Rightarrow \ ABA \ BBA \ ABA \ \ BBBBBBBBBBBABA \]

Hint: draw a straight line.
Cantor Dust

How much of the line is removed?

\[
A = \text{line} \\
B = \text{no line}
\]

\[
\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}
\]

\[
\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{2}{3}} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1 = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1}
\]

\[
\leadsto \frac{2}{9} < \frac{1}{4} < \frac{1}{3}
\]

What about the number \( \frac{1}{4} \) for instance?
What does this sequence do?

1. $X \rightarrow X + YF$
2. $Y \rightarrow FX - Y$

Start with $FX$. 

\[ FX + YF \]
\[ FX + YF + FX - YF \]
\[ FX + YF + FX - YF + FX + YF - FX - YF \]

\[ F + F + F - F + F + F - F - F \]
Dragon Curve

Do the following things:

- $F$ means move forward
- $+$ means turn clockwise 90°
- $-$ means turn counterclockwise 90°
Dragon Curve

Here are some steps:

\[ F + F \]

\[ F + F + F - F \]
Dragon Curve

Here are some steps:
Dragon Curve
Another L-system

Start with:

One rule:

\[ F \rightarrow F + F - F + F \]
Another L-system

Start with:

\[ F \rightarrow -F - F \]

One rule:

\[ F \rightarrow F + F - -F + F \]

Let:

- \( F \): move forward
- \( + \): move clockwise by \( \frac{\pi}{3} \)
- \( - \): move counterclockwise by \( \frac{\pi}{3} \)
Koch Snowflake
What does this sequence do?

1. 0 → 01
2. 1 → 10

Start 0.

⇒ 01
⇒ 0110
⇒ 01101001

⇒ 0110100110010110
Prouhet-Thue-Morse Sequence

It does lots of things.

Call the sequence $t = t_0 t_1 t_2 \ldots = 0 \, 1 \, 1 \, 0 \, 1 \, 0 \, 0 \, 1 \, \ldots$
Draw a picture

- If $t_n = 0$, turn by $\pi$
- If $t_n = 1$, move ahead one unit and then rotate counterclockwise by an angle of $\frac{\pi}{3}$. 
Koch Curve
Another definition of the PTM Sequence

Write out all integers in base-2 (i.e., binary).

Define $s_2(x)$ to the sum of the digits in the base-2 representation of $x$. 

\[
\begin{array}{cccccccccccc}
0 & 1 & 10 & 11 & 100 & 101 & 110 & 111 & 1000 & 1001 \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
0 & 1 & 1 & 0 & 1 & 2 & 2 & 3 & 1 & 2 \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
0 & 0 & 1 & 0 & 0
\end{array}
\]
Yet another definition of the PTM Sequence

Write the number, then flip the bits and rewrite it.

Define:

\[ X_0 = 0 \]
\[ X_{n+1} = X_n \bar{X}_n \]

where \( \bar{x} \) means change all 0's to 1's, and 1's to 0's.
Equivalent definitions

Prove that the last two definitions are equivalent. In other words, prove:

\[ t_n = s_2(n) \mod 2 \]

**Proof:** Use induction.

Base case: \( n = 0 \).

Assume true for all \( n < n' \).

Let \( k \) be the integer such that \( 2^k \leq n < 2^{k+1} \).
A *morphism* is a map $h$ on strings that satisfies the identity

$$h(xy) = h(x)h(y)$$

for all strings $x, y$. 
PTM morphism

Define the PTM morphism

\[ \mu(0) = 01, \]

\[ \mu(1) = 10. \]

Then

\[ \mu(0) = 01 \]
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\[ \mu^3(0) = 01101001 \]
\[ \mu^4(0) = 0110100110010110 \]
Fixed point

The fixed point of a morphism $h$ beginning with $a$ is:

$$h^\omega(a) = \lim_{m \to \infty} h^m(a).$$
Equivalent definition

Let’s prove $\mu^n(0) = X_n$.

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Prove true for $n = k + 1$.

$$\mu^{k+1}(0) =$$
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\begin{align*}
\mu^{k+1}(0) &= \mu^k(\mu(0)) \\
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= X_k \bar{X}_k \\
= X_{k+1}
$$
Overlap-free

A *square* is a word of the form $xx$.

A *cube* is a word of the form $xxx$.

An *overlap* is a word $axaxa$, with a single letter $a$ and $x$ being any word.

English word examples?
Overlap-free

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English word examples?

How does this relate to the Thue-Morse sequence?
What does this sequence converge to?

Consider the sequence:

\[
\begin{align*}
1 & \quad 1/2 & \quad 1/2 \\
2' & \quad 3/4' & \quad 5/6 \\
& \quad 7/8 & \quad 9/10
\end{align*}
\]

What does this converge to?
Proof

First, observe the limit is:

\[ \Pi_{n \geq 0} \left( \frac{2n + 1}{2n + 2} \right)^{(-1)^{tn}} \]

Let:

\[ P = \Pi_{n \geq 0} \left( \frac{2n + 1}{2n + 2} \right)^{(-1)^{tn}} \]

\[ Q = \Pi_{n \geq 0} \left( \frac{2n}{2n + 1} \right)^{(-1)^{tn}} \]

Notice that

\[ PQ = \frac{1}{2} \prod_{n \geq 1} \left( \frac{n}{n + 1} \right)^{(-1)^{tn}} \]
Proof

\[ PQ = \frac{1}{2} \prod_{n \geq 1} \left( \frac{n}{n + 1} \right)^{(-1)^{tn}} \]

Rewrite, breaking over odd and even indices:

\[ PQ = \frac{1}{2} \prod_{n \geq 0} \left( \frac{2n + 1}{2n + 2} \right)^{(-1)^{t2n+1}} \prod_{n \geq 1} \left( \frac{2n}{2n + 1} \right)^{(-1)^{tn}} \]

\[ = \frac{1}{2} P^{-1} Q. \]

Thus, \( P^2 = \frac{1}{2} \).
Summary

- recursion is very natural
- proving results using recursion is powerful
- recursion has a self-contained, succinct form
- recursion is beautiful

\[ \text{math} = \text{recursion} \]