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Grade 6 Math Circles

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Algebra - Solving Equations

Today we are going to be looking at solving equations. Solving an equation just means finding out what number a variable like, x, equals to make an equation like, $x + 3 = 5$, work. Knowing how to solve equations can be useful in areas like:

- Saving and investing money: [$100(1 - x) = 95$] where x is an interest rate.
- Construction: [$8 * x = 104$] where x is the length of a pool.
- Sales: [$20x - 50 = 150$] where x is the amount of merchandise sold.

Solving with Addition and Subtraction

We'll start with simple addition and subtraction examples, as we must understand how these work before we can move on to more difficult problems.

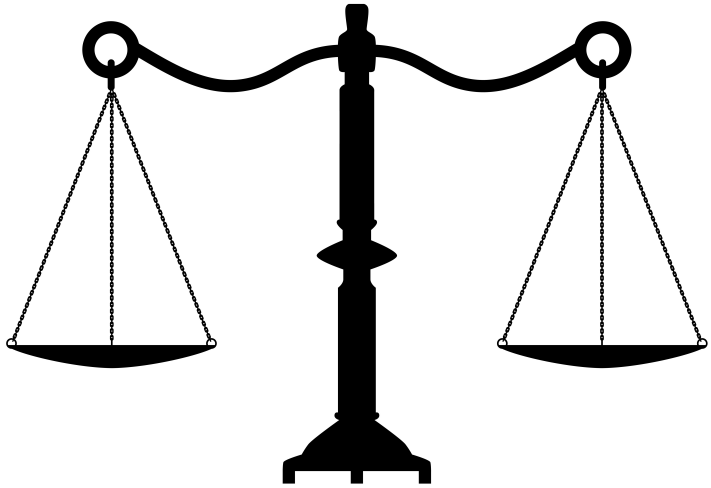
$$x + 6 = 12$$

$$10 - y = 2$$

$$z + 6 - 4 = 10$$

Notice that variables can be any letter you choose.

The best way to understand what we are doing when we "solve" an equation is to visualize a classic scale:



$$x + 6 \qquad = \qquad 12$$

The left platform represents the left side of an equation, and the right platform represents the right side of an equation. These weights are equal, but what we must find is the value of x that is making this happen. To accomplish this, we must add or take away weight from the left side of the scale until all we are left with is a weight of x. However, we must also add or take away the same weight from the right side of the scale to make sure both sides still weigh the same.

$$x + 6 = 12$$

Looking at the left side of the equation, we see a 6 has been added to x. Let's subtract this 6 from both sides of the equation (or take away 6 from both sides of the scale).

$$x + 6 - 6 = 12 - 6$$

$$x = 6$$

Both sides of the scale still weigh the same, since we took the same weight from both sides, telling us that x must be equal to 6.

Exercise: Solve the following equations:

$$x - 4 = 15$$

$$y + 5 = 7$$

$$10 + 5 + z = 20$$

- $x - 4 = 15 \;\rightarrow\; x - 4 + 4 = 15 + 4 \;\rightarrow\; x = 19$
- $y + 5 = 7 \;\rightarrow\; y + 5 - 5 = 7 - 5 \;\rightarrow\; y = 2$
- $10+5+z = 20 \;\rightarrow\; 15+z = 20 \;\rightarrow\; 15+z-15 = 20-15 \;\rightarrow\; z = 5$

Solving Equations with Multiplication and Division

Similar to the addition and subtraction case, to solve an equation that involves multiplication and division, we must get x all on its own. Here are some examples:

$$3x = 27$$

$$5 * y = 5$$

$$\frac{x}{4} = 4$$

$$\frac{10}{y} = 5$$

What we must understand before we can solve these equations is that division is the reverse operation of multiplication and vice versa. If x is being multiplied by a number, to undo this operation we must divide by that same number.

Here is an illustration:

Let's start with the number "1" and multiply it by some other numbers:

$$\mathbf{1} \rightarrow \mathbf{1} * 2 * 3 * 4 * 5$$

Now if we want to get back to our original "1," all we have to do is divide by those same numbers.

$$\frac{\mathbf{1} * 2 * 3 * 4 * 5}{5} = \mathbf{1} * 2 * 3 * 4 \rightarrow \frac{\mathbf{1} * 2 * 3 * 4}{4} = \mathbf{1} * 2 * 3$$
$$\frac{\mathbf{1} * 2 * 3}{3} = \mathbf{1} * 2 \rightarrow \frac{\mathbf{1} * 2}{2} = \mathbf{1}$$

We use this same logic when a variable like x is multiplied by any number.

Exercise: Solve the following equations:

$$5x = 20$$
$$\frac{y}{5} = 20$$
$$\frac{3z}{4} = 6$$

- $5x = 20 \;\rightarrow\; \frac{5x}{5} = \frac{20}{5} \;\rightarrow\; x = 4$
- $\frac{y}{5} = 20 \;\rightarrow\; 5 * \frac{y}{5} = 5 * 20 \;\rightarrow\; \frac{5y}{5} = 100 \;\rightarrow\; y = 100$
- $\frac{3z}{4} = 6 \;\rightarrow\; 4 * \frac{3z}{4} = 4 * 6 \;\rightarrow\; \frac{4(3z)}{4} = 24$

  $\rightarrow 3z = 24 \;\rightarrow\; \frac{3z}{3} = \frac{24}{3} \;\rightarrow\; z = 8$

Cross Multiplication

When we come across equations with our variable in the denominator position of a fraction, using cross multiplication to solve for x is very useful:

$$\frac{4}{x} = \frac{1}{2}$$

What we mean by "Cross Multiplication," is taking the numerator of the left side, 4, and multiplying it by the denominator on the right side, 2. This then becomes our new left side of the equation. We then multiply the numerator of the right side, 1, by the denominator of the left side, x, and this becomes our new right side of the equation. As an illustration, draw a line to join the two pairs of numbers we are multiplying below:

$$\frac{4}{x} = \frac{1}{2}$$

This equation then becomes:

$$8 = x \text{ or } x = 8$$

To check our answer, let's plug in $x = 8$ into the original equation to see if it works out properly:

$$\frac{4}{x} = \frac{4}{8} = \frac{1}{2}$$

Exercise: Solve the following equations using cross multiplication.

$$\frac{3}{x} = \frac{1}{3} \;\rightarrow\; 9 = x \;\rightarrow\; x = 9$$

$$\frac{15}{x} = \frac{5}{1} \;\rightarrow\; 15 = 5x \;\rightarrow\; \frac{15}{5} = \frac{5x}{5} \;\rightarrow\; 3 = x \;\rightarrow\; x = 3$$

Collecting Like Terms

When we say "Collecting Like Terms," we mean adding or subtracting the same variables together, and doing the same for the numerical values. We then want to put all our variables on one side of the equation and all of our numerical values on the other. This makes our equations look nicer and will help us in solving them. Here is an example to illustrate:

$$x + x + y + y = 1 + 1 + 2 + 2$$
$$2(x) + 2(y) = 2(1) + 2(2)$$

We treat variables the same way we treat regular numbers. Looking at the right side of the equation, we can say there are two "1s" and two "2s," (instead of adding the numbers right away) just like the left side for x and y. We know that 1 + 1 is the same as 2 * 1, and 2 + 2 is the same as $2 * 2$. Using this same logic, $x + x$ is the same 2 * x ($2x$), and $y + y$ is the same as 2 * y ($2y$):

$$2x + 2y = 2 + 4 \rightarrow 2x + 2y = 6$$

Exercise: For each equation, collect all like terms.

$x + x - x = 5 - 3 \;\rightarrow\; 2x - x = 2 \;\rightarrow\; x = 2$

$2x + 3 - 2 - x + y - y = 1 \;\rightarrow\; x + 1 + 0 = 1 \;\rightarrow\; x + 1 - 1 = 1 - 1 \;\rightarrow\; x = 0$

$x + y + 4 - 2 = 5 - x \;\rightarrow\; x + y + 2 = 5 - x \;\rightarrow\; x + y + 2 - 2 = 5 - x - 2 \;\rightarrow\; x + y = 3 - x \;\rightarrow$
$x + y + x = 3 - x + x \;\rightarrow\; 2x + y = 3$

Putting it all together

Now that we have looked at examples involving, for the most part, one type of operation, let's look at some examples that involve more than one operation:

$$\frac{3x+5}{2} = 10$$

Remember, what we are trying to do is isolate for x (get x all on its own). When there is more than one operation, we must undo these operations in the reverse order in which they were done to x. First, let's find out the order in which the above operations were performed:

$$x \rightarrow 3x \rightarrow 3x+5 \rightarrow \frac{3x+5}{2}$$

Now that we see the order in which these operations were applied, all we have to do is trace our steps back to our x by performing the corresponding reverse operations. Also, keep in mind that we must perform these reverse operations to the right side of the equation,"10," as well:

$$\frac{3x+5}{2} \rightarrow 3x+5 \rightarrow 3x \rightarrow x$$

$$10 \rightarrow 20 \rightarrow 15 \rightarrow 5$$

Where our chain of reverse operations ends, we can see that $x = 5$.

Word Problems

So far we have seen examples that tell you to solve an equation, but now we will move onto word problems where the equation we want to solve is not given. Let's look at an example as an illustration:

Example: You went grocery shopping yesterday and spent \$30. When you went to check how you spent this much, you realized the price for chicken breasts had ripped off of the receipt. If you spent \$5 on pizza, \$7 on cereal, \$4 on vegetables, how much did you spend on each chicken breast, knowing that you bought two of them?

We start off word problems by defining what our variables are going to represent. To do this, we do this with a "Let" statement:

Let c be the cost of one chicken breast. We then have the following algebraic equation:

$$2c + 5 + 7 + 4 = 30$$

Now all we have to do is solve for c:

$$2c + 5 + 7 + 4 = 30$$

$$2c + 16 = 30$$

$$2c + 16 - 16 = 30 - 16$$

$$2c = 14$$

$$\frac{2c}{2} = \frac{14}{2}$$

$$c = 7$$

Therefore, you spent 7\$ on each chicken breast.

Problem Set

"*" indicates challenge question

1. Solve the following equations:

   (a) $x + 4 = 10$

   (b) $x - 2 = 0$

   (c) $10 - y = 2$

   (d) $6 + 10 - 4 + z = 12$

   (e) $x + 5 = 15, \quad y + x = 12, \quad 10 - 5 + x - y + 6 + z = 20$

2. Tim had a dozen eggs in the fridge one night. When he went to make breakfast the next morning, there were only half of a dozen eggs left. How many eggs mysteriously went missing? Start your answer by representing this problem with an algebraic equation.

3.
   - What is a variable?
   - What is the reverse operation of subtraction; what is the reverse operation of multiplication?

4. Collect the like terms on both sides of this equation. You do not have to solve this equation:

$$x + 2y + 3 + 8 + 3x - y = z + 5 - 1 + 2z + 5 - z$$

5. There is an assortment of 10 flowers with roses, tulips, and daisies. There must be exactly 6 roses in the assortment, at least 2 tulips, and any amount of daisies. What are all the possible arrangements?

6. Husky is 6 years older than Colly, and Colly is 3 years younger than Poodle. If Husky is 19 years old, how old is Poodle? Represent this question with an algebraic equation.

7. Two consecutive numbers have a sum of 25. What are these two numbers? Represent this problem with an algebraic equation.

8. For the equation, $\frac{20}{x} = 5$, represent the right side as a fraction and then use cross multiplication to solve for x.

9. Solve the following equations

   (a) $5x = 25$

   (b) $9y = 36$

   (c) $\frac{x}{6} = 4$

   (d) $\frac{16}{x} = 8$

   (e) $\frac{3y}{5} = 9$

   (f) $\frac{z}{6} = \frac{1}{3}$

10. Mr. Johnson was printing out test papers for his class, but realized he accidentally printed 3 times the amount he actually needed. If he printed out 63 test papers, what is the size of Mr. Johnson's classroom (assuming each student gets one test paper).

11. Solve for each variable in this system of equations:

    - z + 2x - y = 10
    - 6y + x - 4 = 11
    - 3x + 1 = 10

12. A classroom of 50 students is divided into two groups, with one of the groups having 8 students more than the other. What is the size of each group?

13. * Solve the following equations

    (a) $\frac{x-1}{5} = 1$

    (b) $\frac{4x-2}{2} = 7$

    (c) $\frac{x+2-1+x}{3} = 3$

14. * The area of a pig pen is 1000m^2, with the width being 20m.

    (a) What is the length of the pig pen in metres?

    (b) If a pig is 2 metres in length and a total of 500 pigs can fit in the pen exactly, what is the width of each pig?

15. * \$50 is to be split among three friends. Friend **B** gets double the amount of friend **A**, and friend **C** gets \$10 more than friend **B**. How much money does each friend get?

16. * Two consecutive, odd numbers have a sum of 52. What are these two numbers. Represent this problem with an algebraic equation (Hint: any odd number can be written as $2n + 1$).

17. * Solve the following equations

    (a) $x^2 = 4$

    (b) $x^3 = 8$

    (c) $(x + 1)(x - 3) = 0$