LINEAR RECURRENCE RELATIONS: HANDOUT 1

This handout contains 7 exercises, each with many parts. I will alternate speaking and having you do the exercises. My goal is for you to discover things on your own, with a little push here and there. I will let you know when to try certain exercises. If you finish quickly, you can move on to the next exercises, but my rule is that you are only allowed to ask me questions about exercises that I have so far assigned. Exercise 1 is intended to be a chance to brush up on some skills you will need throughout the three sessions. You don't need to do them if you are completely comfortable with the concepts, but if you aren't, you may want to try them as homework before the second session.

Exercise 1. This is a variety of exercises. You should at least read through them to see that you know how to do them. The skills will be needed in the last two sessions. They are not in any particular order. In particular, the later ones are not necessarily harder than earlier ones.

(1) Find roots of the following quadratic equations by any means you like. They may not have real roots, but that's ok. Expressions with things like $\sqrt{-5}$ are ok.
   (a) $x^2 - 3x + 2 = 0$
   (b) $x^2 + x - 1 = 0$
   (c) $x^2 + 4x - 32 = 0$
   (d) $x^2 + 2x - 1 = 0$
   (e) $x^2 - 2x - 1 = 0$

(2) Solve the following systems of equations.
   (a) $x + y = 7$ and $2x - y = 9$
   (b) $x + 3y = \sqrt{2}$ and $x - y = \frac{1}{2}$
   (c) $2x + y = 1$ and $-x + y = 10$
   (d) $4x + 3y = 12$ and $-7x + 5y = 4$

(3) Simplify the following expressions.
   (a) $(2 + \sqrt{3})^4$
   (b) $(1 + i)^2$ (remember, $i^2 = -1$)
   (c) $(3 - \sqrt{2})^3$
   (d) $\frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2}$

Exercise 2. Compute the first 8 or 10 terms of the following recurrence relations. Try to avoid using a calculator as much as possible.

(1) $a_n = a_{n-1} + 3a_{n-2}$ with $a_0 = 2$ and $a_1 = 1$
(2) $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 3$ and $a_1 = 2$
(3) $a_n = 2a_{n-1} - a_{n-2}$ with $a_0 = 1$ and $a_1 = 1$
(4) $a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 3$
(5) $a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$
(6) $a_n = -a_{n-1} + 6a_{n-2}$ with $a_0 = 1$ and $a_1 = 1$
(7) $a_n = a_{n-1} - 2a_{n-2}$ with $a_0 = 3$ and $a_1 = 2$
(8) $a_n = -a_{n-1} - a_{n-2}$ with $a_0 = 4$ and $a_1 = 7$
(9) $a_n = 2a_{n-1} - 3a_{n-2}$ with $a_0 = 1$ and $a_1 = 1$
(10) $a_n = 2a_{n-1} + a_{n-2}$ with $a_0 = 1$ and $a_1 = 1 + \sqrt{2}$ (Bonus: Compute $(1+\sqrt{2})^n$ for $n = 1, 2, 3, 4$)

Exercise 3. Compute the first few terms of the sequence for each recurrence relation at the different starting values. Try writing them in a table.

(1) $a_n = a_{n-1} + a_{n-2}$
  (a) $a_0 = 1, a_1 = 2$
  (b) $a_0 = 2, a_1 = 2$
  (c) $a_0 = 2, a_1 = 4$
  (d) $a_0 = 3, a_1 = 4$
  (e) $a_0 = -1, a_1 = 0$
  (f) $a_0 = 4, a_2 = 6$
  (g) Look for connections between
    (i) (a) and (c)
    (ii) (a), (b), and (d)
    (iii) (a), (b), and (e)
    (iv) (a), (b), and (f)
(2) $a_n = 3a_{n-1} - 2a_{n-2}$
  (a) $a_0 = 1, a_1 = 1$
  (b) $a_0 = 1, a_1 = 2$
  (c) $a_0 = -1, a_2 = -2$
  (d) $a_0 = 2, a_1 = 3$
  (e) $a_0 = -1, a_1 = 0$
  (f) Look for connections between

(i) (b) and (c)
(ii) (a), (b), and (d)
(iii) (a), (b), and (e)

Exercise 4. Compute the sequence $c_n = 3c_{n-1} - 2c_{n-2}$ (from Exercise 3 [2]) with $c_0 = 3$ and $c_1 = 1$ using ideas from the previous exercise. (Hint: $3 = 5 \cdot 1 - 2 \cdot 1$ and $1 = 5 \cdot 1 - 2 \cdot 2$)

Exercise 5. Find a closed formula for c_n from Exercise 4. That is, a formula that doesn't refer to the previous terms. Of course, it will depend on n.

Exercise 6. This exercise deals with the sequence $a_n = 5a_{n-1} + -6a_{n-2}$.

(1) Compute the sequence for $a_0 = 1$ and $a_1 = 3$.
(2) Compute the sequence for $a_0 = 1$ and $a_1 = 2$.
(3) Give a closed form for the sequences in (1) and (2).
(4) Find a closed form for the sequence when $a_0 = 0$ and $a_1 = 1$.
(5) Do (4) with $a_0 = -1$ and $a_1 = 0$.
(6) Pick any starting values, a_0 and a_1, and find a closed form for the sequence.

Exercise 7.

(1) Consider the recurrence $a_n = 6a_{n-1} - 8a_{n-2}$. Find all values r so that when $a_0 = 1$ and $a_1 = r$ then $a_n = r^n$ for all n.
(2) Do (1) where $a_n = 2a_{n-1} + a_{n-2}$ (Hint: don't expect integer values for r).
(3) Find a closed form for the recurrence relation in (1) with $a_0 = 2$ and $a_1 = 7$.
(4) Find a closed form for the recurrence relation in (2) with $a_0 = 2$ and $a_1 = 1$. Compute terms a_{10}, a_{11}, a_{12}, and a_{13} for this sequence, using a calculator. Compute a_{11}/a_{10}, a_{12}/a_{11} and a_{13}/a_{12} What do you notice?