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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

February 25/26, 2014

Unique Geometry

Dudeney's dissection

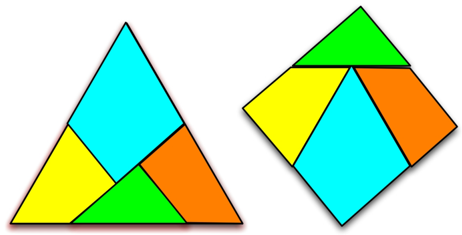


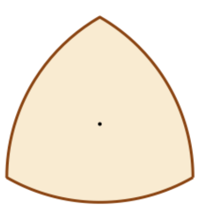
Image from oeis.org

Dudeney's dissection is a method to switch a shape between a triangle and a square. There are 4 pieces that are connected together using hinges. One place this is useful is rearranging tables at a restaurant. (See `https://www.youtube.com/watch?v=yc_bp5B-MWs`)

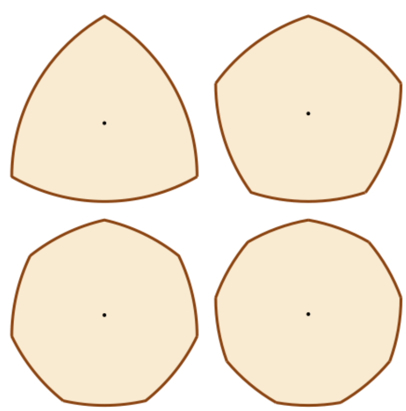
Shapes of Constant Width

A circle is special because the diameter is the same all the way around the circle. The **diameter** is a straight line drawn from one point on the circle to another, through the center.

However, this feature does not make a circle unique. **Shapes of constant width** all have this property. The simplest example of such a shape is a Reulaux triangle.



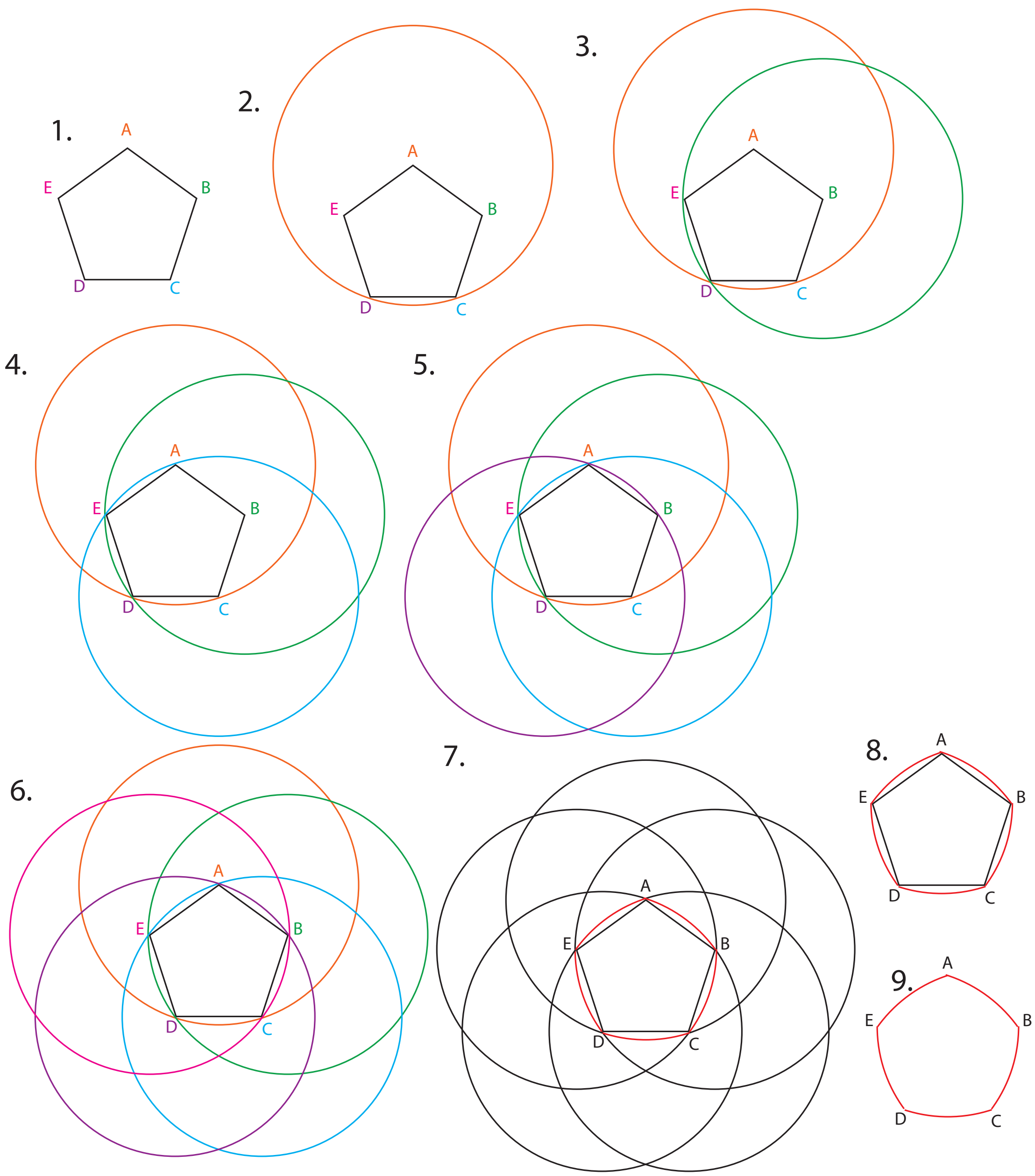
This shape is called a Reuleaux triangle.



Shapes of constant width can have different numbers of sides.

Construction of a Shape of Constant Width

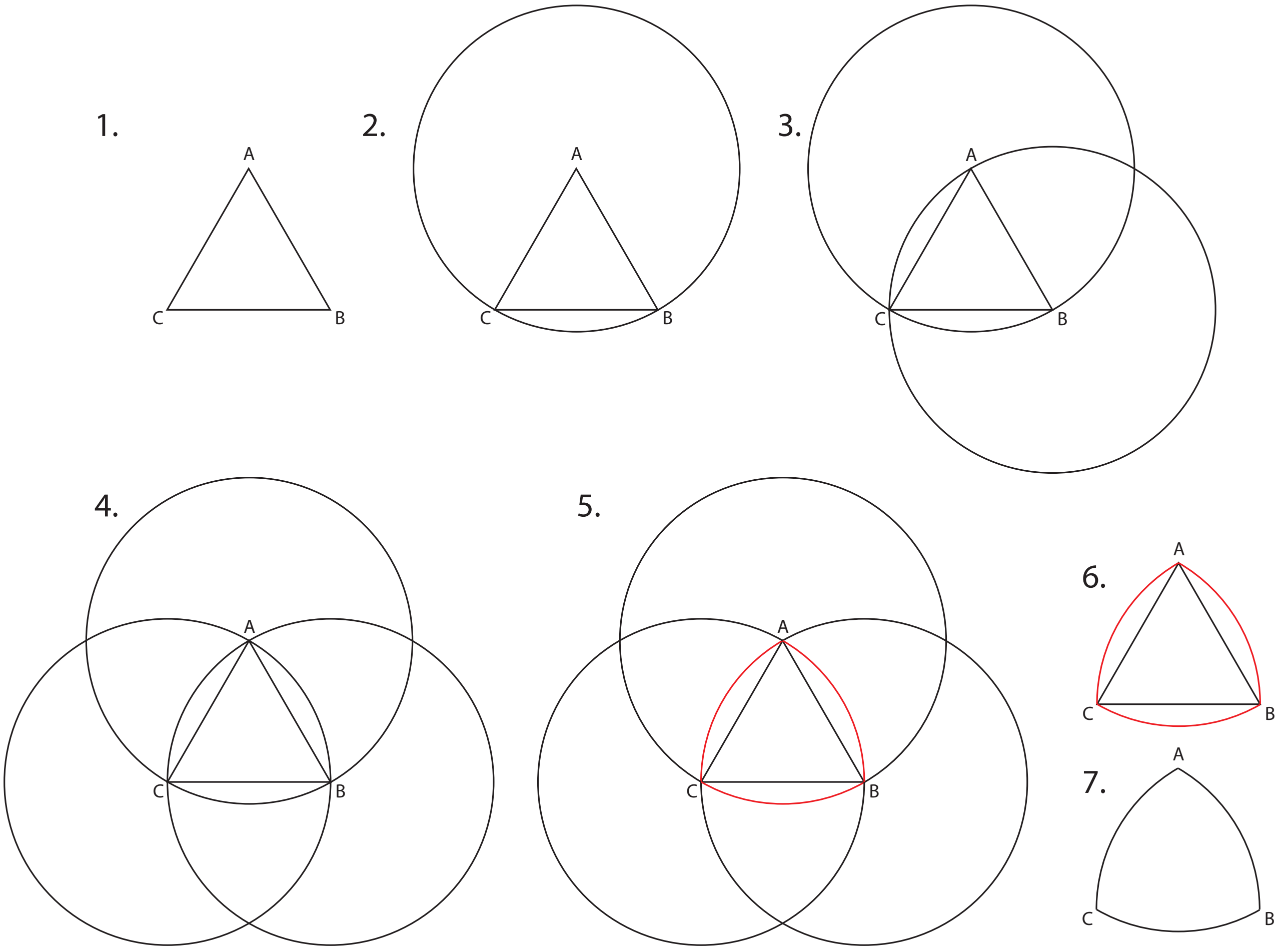
1. Start with a regular shape with an odd number of sides, like a triangle or a pentagon.
2. Draw an arc (part of a circle) between two of the points by drawing a circle centred at one of the points that intersects with the two points opposite it.
3. Draw an arc around the remaining points.
4. Erase everything except the arcs.



Exercise 1

1. What shape would you use to construct a shape of constant width with 3 sides?
   Triangle

2. Draw the shape of constant width with 3 sides.

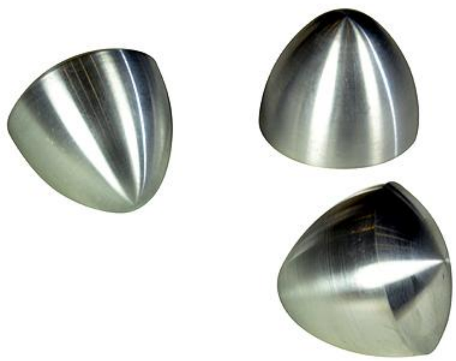


3. Where have you seen a shape like this before? See "Applications" below

Solids of Constant Width

Try this: You will need some spheres like a small ball or marbles and a slightly heavy flat object like a book with a hard cover. Put the book on top of the spheres and roll the book around. Notice how the book stays flat no matter how you roll it.

The same is true for solids of constant width . Solids of constant width are similar to shapes of constant width, but they are 3-dimensional. (See `https://www.youtube.com/watch?v=jYf3nOYM_mQ`)



These are solids of constant width.
(pictures from www.grand-illusions.com)

This is a book on top of 3 solids of constant width. Notice how the book is flat even though the solids are all on different sides.

Applications

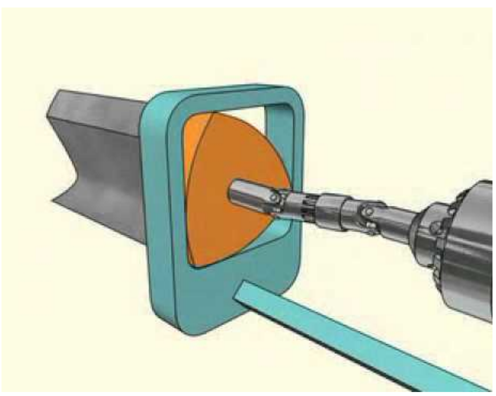
Shapes of constant width have many of the same applications as circles. For example, you can make a bike that has shapes of constant width as its tires. You can also make a manhole cover that has this shape, but won't fall through. Unlike a circle, this cover would have to be placed in a certain way to fit. Another application is coins. Coins have to have a constant width so that machines can detect them properly. For example, if a coin didn't have constant width it might get stuck in a vending machine. Of course, we could just make all coins circular, but that's no fun! In Canada, the loonie is a shape of constant width with 11 sides.

Drilling a square hole

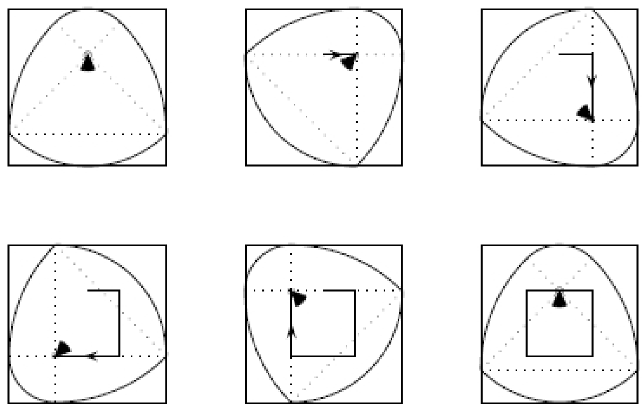
Most (if not all) commonly found drills will make a circular hole. This is because it is the easiest shape to drill. You can easily drill a circular hole with a round drill bit. But what if you wanted a square hole? You can try using a square drill bit, but that will still drill a circle because the square is moving in a circular motion.

However, we can drill a square hole by using shapes of constant width. Instead of using a cylinder or rod, which will rotate in a circle, we can use a shape of constant width, which will rotate in a square.

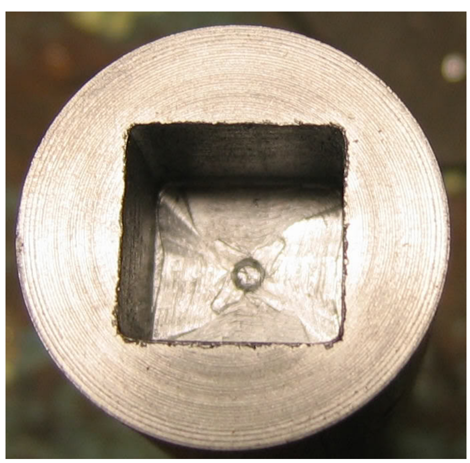
(See `https://www.youtube.com/watch?v=S1JH19-JwzA`
and
`https://www.youtube.com/watch?v=UBQFdmPiym8` (from 0:47))



The drill looks similar to this.
(from youtube.com)



This is the square being drilled.
(from web.mat.bham.ac.uk)



Here is the square hole.
(bbs.homeshopmachinist.net)

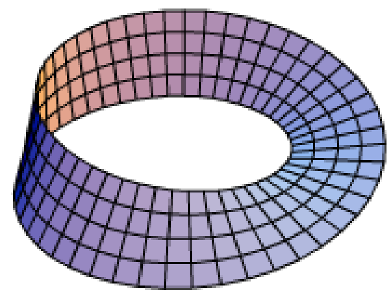
Mobius Bands

A **Mobius band** or **Mobius strip** is a unique stip that has some interesting properties. It

is named after the German mathematician and astronomer August Ferdinand Mobius. He and mathematician Johan Benedict Listing both discovered the band separately.

How to make a Mobius band

Take a strip of paper and twist one of the ends so it is flipped over, now tape the two ends together. Be sure to tape all the way across the edge, or else your band will come apart later.



`http://commons.wikimedia.org/wiki/`

Exercise 2

1. How many sides does a Mobius band have? To find out, draw a line along the band until you can't go any further or you get back to where you started. Make sure not to lift you pencil of the band. Since we are able to draw a line all around the band without lifting the pencil, there is only one side. To see what happens when there is two sides, you can make a regular loop. Notice that when you draw a line in the same manner it does not cover both sides of the paper. You have to draw a total of 2 lines, so there are two sides.

2. How many edges does a Mobius band have? To find out, use a marker to colour the edge of the band until you get back to the beginning. Do this until all the edges are coloured. If you have to lift your marker then it is a separate edge. Similar to the number of sides, there is only one edge. Try it with a normal loop to see what it looks like with 2 edges.

Investigating the Mobius band

Experiment 1

What do you think will happen if you cut the band in half (on the long side, without cutting an edge)?

Try it and see what you get. Make sure you begin cutting by making a small hole in the middle of your band, and not by cutting into the edge.

1. Was your hypothesis correct?

2. What actually happened?

Note that you should be cutting a path similar to the one you drew in Exercise 2. Cutting the band in half will result in a single long twisted loop.

Experiment 2

Make a hypothesis about what will happen if you cut in the same way as Experiment 1, but in thirds instead of half.

1. Were you right?

2. What actually happened?

Making sure that you cut around the Mobius band twice, you will get two twisted loops that are linked together.

Experiment 3

Do the properties of the Mobius band stay the same if we construct it with more than one half-twist? one with two half-twists, one with three, and one with four. Then complete the following table:

Number of half-twists	Number of sides	Number of edges	What happens when cut in half
2	2	2	2 linked loops
3	1	1	One knotted loop
4	2	2	Two linked loops

Write down any patterns that you see:

Loops with an odd number of twists have 1 side and 1 edge and create one loop when cut in half. Loops with an even number of twists have 2 sides and 2 edges and create two loops when cut in half.

Applications of the Mobius band

Mobius bands have been used as conveyor belts that last longer because the entire surface area of the belt gets the same amount of wear, and as continuous-loop recording tapes to double the playing time without having to flip over the tape.

Mobius bands are common in the manufacture of fabric computer printer and typewriter ribbons, as they allow the ribbon to be twice as wide as the print head while using both half-edges evenly.

Exercises

1. **Mobius' Crosses:** To begin, take two strips of paper and tape them together at a right angle so they make a "+". Be sure to securely tape all of the edges. You will need three of these crosses altogether.

   (a) With your first cross, take two opposite arms and tape them together in an untwisted loop. Take the other two arms and tape them together in an untwisted loop. Your cross should now look like a figure-8 that is half twisted in the middle. What do you think will happen when you cut both loops down their middles? Try it to test your hypothesis.

   (b) Using your second cross, again tape opposite ends into loops, but this time make one loop and one Mobius band. What do you think will happen when we cut these loops down their middles? Test your hypothesis.

   (c) With your third cross, tape both pairs of opposite ends into Mobius bands. Make a hypothesis as to what will happen when they are cut in half and test your hypothesis (Note: depending on how you twisted your loops, a couple of different outcomes can happen, don't worry if your cross turns out different than your friend's).

2. **Double Mobius Bands:** Hold two strips of paper together in your hand (one underneath the other), and bring the ends together with a half-twist (like you are making a Mobius band). Get a friend to help you tape the top two strips together and the bottom two strips together. You should now have two Mobius bands nestled snugly side by side.

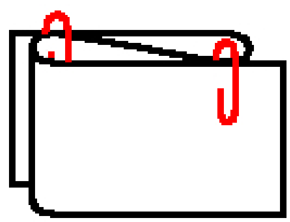
   (a) Take a pencil and put it between the two loops. Move the pencil around the loop one time until it returns to the place where you began. Is your pencil facing the same or opposite direction than it was before? Can you explain this? What happens when you move your pencil around the loop a second time?

   (b) Make a hypothesis as to what will happen when we pull the two bands apart. Test your hypothesis.

3. Take a small piece of paper (at most one quarter of a full sheet) and in the middle trace around the rim of a penny. Then carefully and accurately cut a circular hole in your paper along that outline. Your challenge now is to pass a quarter through this hole without tearing the paper.

4. For this puzzle, use half of a piece of paper. Your challenge is to cut a hole in this paper so large that a person can fit through it.
   Hint: Cutting a hole does not necessarily mean we have to remove parts of the paper.

5. Using a strip of paper and two paperclips, fold the strip over and then back, so you get a "Z" shape, like the picture below. Now slide the paperclips on from one side, so that the paper is held in this shape.
   (a) Make a hypothesis as to how far apart the paperclips will land when you pull hard on both ends of the paper, making them fly into the air. Try it out, are you surprised by your results?
   (b) Try to achieve the same result using three or more paperclips, and more than two folds.



6. Try to recreate the figure below using a single sheet of paper and no tape.