UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

February 18/19, 2014

Graph Theory II

Review

- A **vertex** (plural: **vertices**) is a point that is usually labelled with a letter, number, or other title. (ex. A)
- An **edge** is a line that joins two vertices and is named in the format $\{-,-\}$, where " $-$ " is replaced by the vertices it is incident to. (ex. $\{A, B\}$)
- Two vertices are **adjacent** if there is an edge joining them.
- A vertex and edge are **incident** if they are joined.
- A **walk** is a sequence of vertices where each vertex is adjacent to the vertex before and after it. A walk can be written as "$vertex, vertex, vertex...$". (ex. $A \to B \to E \to A \to C$)
- A **path** is a walk that doesn't repeat vertices.
- A **cycle** is a path that begins and ends at the same vertex.
- Edges **can** be curved or straight (or any other shape).
- Edges **can** cross.
- The placement of vertices **doesn't** matter. Only the connections are important.
- Two edges **can't** connect the same vertices. (ie. There can only be a **maximum** of one edge between any two vertices.)
- An edge **can not** connect a vertex to itself.
- Edges have **no** direction.

Handshake Lemma

The **degree** of a vertex is the number of edges incident to it (remember that this means the number of edges connected to the vertex). The **Handshake Lemma** can be stated as follows:

Theorem. *The sum of the degrees of all the vertices is equal to double the number of edges*

Let's think about why this is. Each edge connects to two vertices. That means that each edge is counted twice when we count the number of connections.

Exercise 1

1. Marcus, Tommy, Chuck, Gerald, and Stacy go to a party. They have never met before so they all shake hands when they get there.

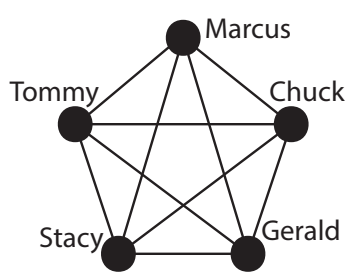
   (a) How many times does each person shake hands? Each person shakes hands 4 times.

   (b) What is the total number of times a hand is shaken? (Note: It counts as two different hands being shaken when two people meet) There are 20 hands shaken.

   (c) How many handshakes take place? 10 handshakes take place.

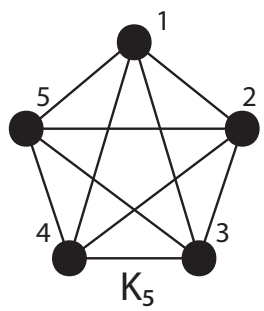
   (d) How do parts (b) and (c) relate? Explain why this makes sense. The total number of times a hand is shaken is double the number of handshakes.

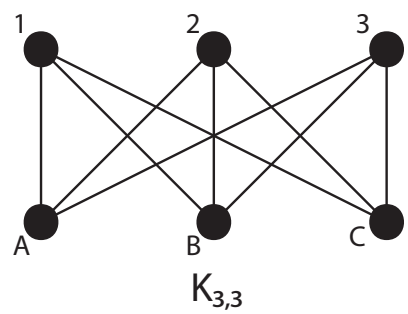
   (e) Draw a graph to represent this situation. What does each vertex represent? What does each edge represent? Each vertex represents a person. Each edge represents a handshake.



Planar Graphs

- A **planar graph** is a graph that can be drawn without any edges crossing. When a planar graph is drawn in its planar form (without any edges crossing), we call it a **planar embedding**.
- An enclosed area bounded by vertices and edges in a planar embedding is called a **face**.
- It may seem like a planar embedding is always possible, but there are some graphs that are impossible to draw without any edges crossing. We call these graphs **non-planar**. There are two notable non-planar graphs we will look at: K_5 and $K_{3,3}$.





Exercise 2

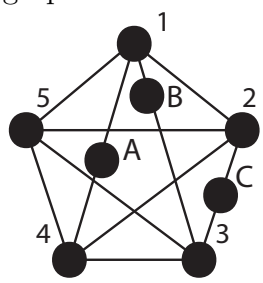
1. Try to draw a planar embedding of K_5 and $K_{3,3}$.

   (a) The **crossing number** of a graph is the least number of edges crossing. Find the crossing numbers of K_5 and $K_{3,3}$. Draw this. Drawing may vary. The crossing number of K_5 and $K_{3,3}$ are 1.

   (b) Explain why the drawing from part (a) can't become planar. Explanations will vary. Students should explain that there is no way to draw the line without crossing.

   (c) If you add edges and vertices to either of the graphs, can it become planar? No, it can not become planar (unless you add a vertex at an intersection of edges). Try to draw it out if you can't visualize it.

- A **subdivision** of a graph is a graph that has vertices added to divide an existing edge. This is a subdivision of K_5:



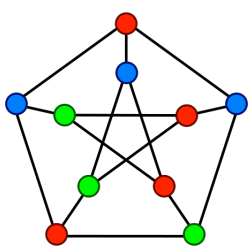
K_5 subdivision

Theorem. *A graph is non-planar if and only if there is a subdivision of K_5 or $K_{3,3}$.*

Note: "If and only if" means that the first statement is true if the second is true, and the second is true if the first is true.

Colouring

A **colouring** is a way of labelling a graph. In a colouring, no two adjacent vertices can have the same colour. We can describe a graph by the minimum number of colours needed. A **k-colouring** (where k is replaced with a number) is a colouring with at most k colours. A graph is **k-colourable** if it has a k-colouring. Here is an example (from `en.wikipedia.org/wiki/Graph_coloring`):



This is the Petersen Graph with a 3-colouring. This is the minimum colouring possible for this graph.

Note: A colouring does not have to use colours. Mathematicians often use numbers instead of colours. With this system, each number represents a colour.

Exercise 3

1. Is a 3-colouring also a 4-colouring? Yes, a 3-colouring is also a 4-colouring.
2. What are some applications of colouring? Colouring can be used in tournaments, maps, scheduling, and puzzles like Sudoku.

Theorem. *Every planar graph is 5-colourable.*

Theorem. *Every planar graph has at least one vertex with a degree of 5 or less.*

Theorem. *Every planar graph is 6-colourable.*

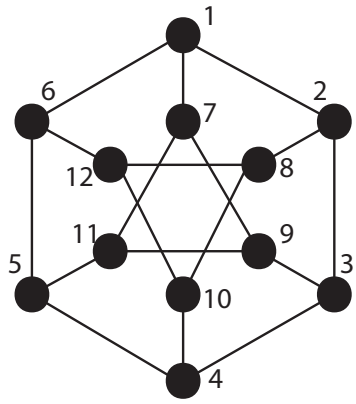
You will prove the last theorem in the problem set.

Problem Set

"*" indicates challenge question

1. Are the following graphs planar? If so, draw a planar representation. If not, explain why. Hint: Only one of these are planar. Try finding K_5 or $K_{3,3}$ subdivisions.

(a)



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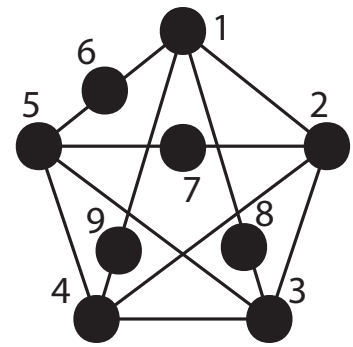

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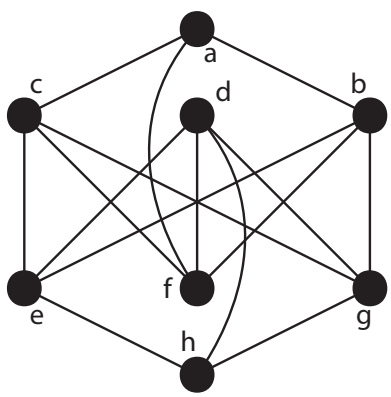

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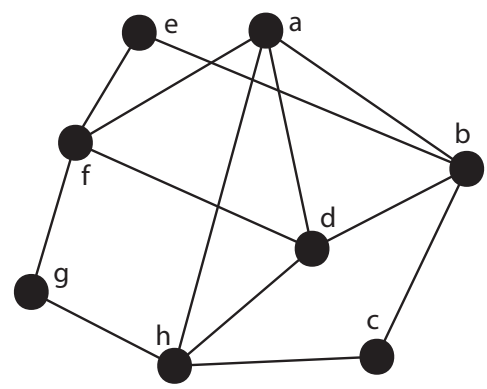
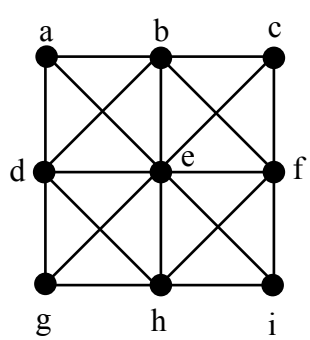
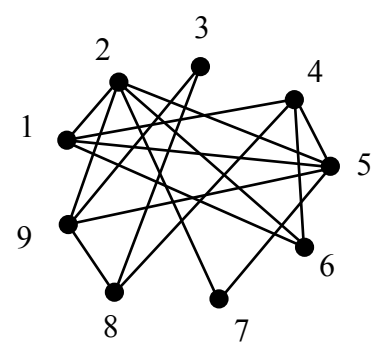


2. Redraw each graph to show it is planar.



(a)



(b)

3. There are five teams in a tournament.

   (a) Use a graph to prove that there is a way for each team to play every other team once.

      i. How many games does each team play?

      ii. What is the total number of games played?

   (b) The organizers of the tournament would like to consider the different number of games each team can play (ie. how many games each team should play). Assuming each team must play at least one game, the options are for each team to play 1 game, 2 games, 3 games, or 4 games. Assume that each team has to play the same number of games, and that teams can't play each other more than once.

      i. Is it possible to draw a graph for all the different number of games? Explain. Be sure to use specific reasons you learnt in this lesson.

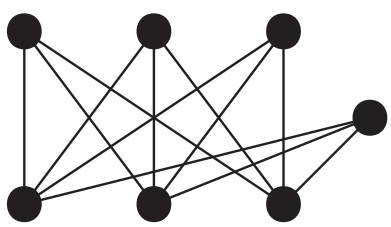
      ii. Draw one graph for each number of games that is possible.

      iii. Can you fix the scenarios from part i) by removing one of the assumptions? You must still assume that each team has to play at least one game.

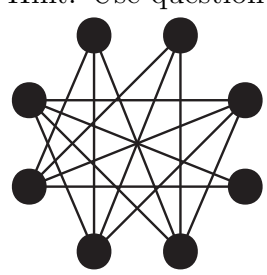
4. *How can you rearrange vertices to make it easier to see the minimum colouring. (Hint: think about grouping vertices according to what they are connected to)

5. Find the minimum colouring for each of the following graphs.

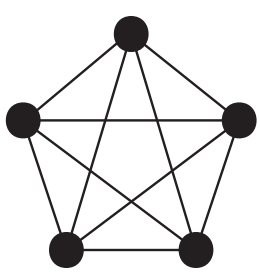
   (a)



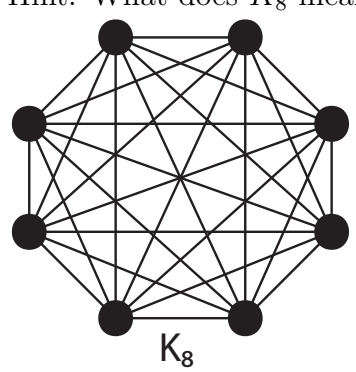
   (b) Hint: Use question 4



   (c)



   (d) Hint: What does K_8 mean?



6. Draw a graph that is 1-colourable, 5-colourable, 10-colourable, 20-colourable, 50-colourable, 100-colourable, and 1000-colourable.

7. Draw a graph that has a minimum colouring of

   (a) 1

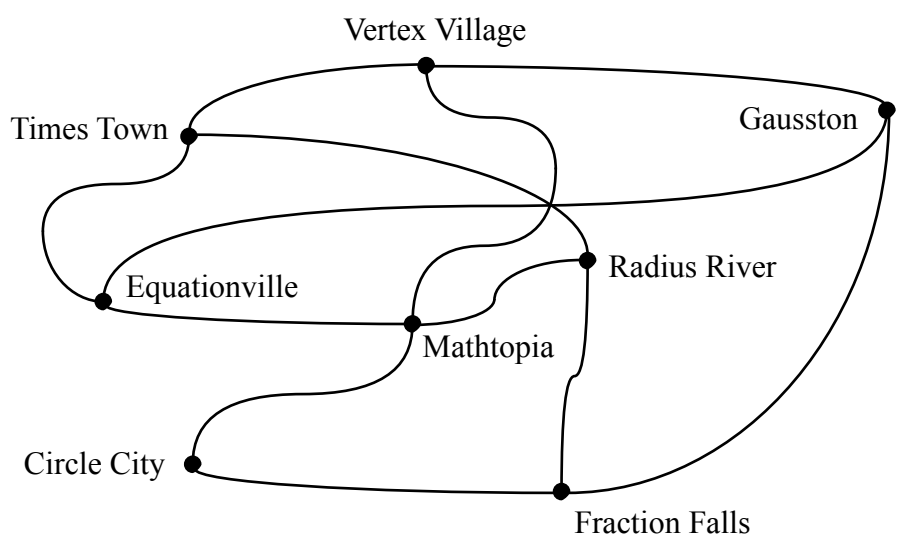
   (b) 3

   (c) 5

   (d) *10

8. A word graph has words as the vertices. Two words are adjacent if they differ by 1 letter. Create a connected word graph that contains the following words: log, top, eat, mud and rag.

9. Euler's Oil is a gas station chain in the country of Mathisfun. The CEO of the company, Leonhard, wants to visit all his gas stations in the country to ensure that each one is stocked with enough fuel. However, he doesn't want to visit any town more than once as he is a very busy man. Suggest a route Leonhard could take from the head office in the capital Mathtopia, through each town exactly once, and return to the capital. Note that despite the intersection of three highways in the centre of the country, there is no way to change from one road to another (i.e. there is no direct road from Mathtopia to Gausston, etc.)



10. Below is a map of South America. Create a graph with the vertices representing countries and the edges joining those countries which share a land border. So Chile and Peru are adjacent, but Peru and Uruguay are not. Use your graph to answer the questions below.

Courtesy of abcteach.com

(a) Find the minimum colouring of the graph, and determine a colour for each country. Watch out for countries that that share a very small border.

(b) When trading goods over land, a $100 tax is paid to each country which the goods travel through. So if Columbia sells coffee to Venezuela, the least amount of tax paid is $100.

   i. What is the least amount of tax paid on wool shipped from Ecuador to Paraguay?

   ii. What country can trade with the most others for exactly $100?

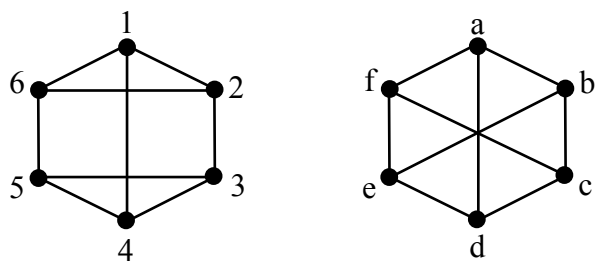
   iii. Brazil raises it's taxes to $400. What is the cheapest amount of taxes paid to ship Venezuelan oil to Paraguay? What about French Guiana to Bolivia?

   iv. To encourage trade, Peru will not tax any goods going though the country. Chile and Suriname want to trade lumber and wheat. What route will result in the least amount of taxes? How much will they pay?

(c) Kevin wants to visit every country in South America on his vacation. He doesn't want to visit any country more than once since crossing the border can take a long time! He will fly into Lima, Peru to begin his trip and fly out of Montevideo, Uruguay. In what order should he visit each country?

11. *Two graphs are **isomorphic** if they are the same graph drawn in a different way. This means that all the connections are the same, and there are exactly the same number of vertices, but the vertices may have different names and the graphs may look different. Determine if the following pairs of graphs are isomorphic. If so, show which vertices match, or explain why not.

(a)



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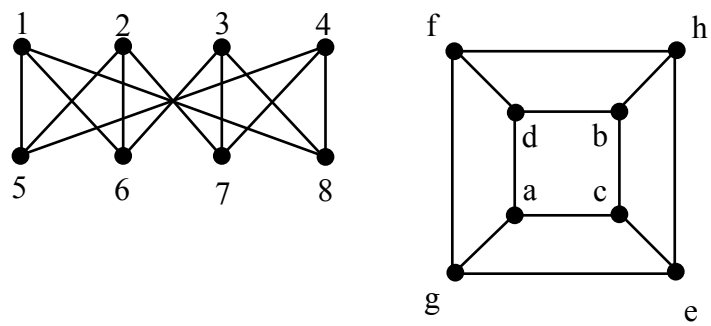
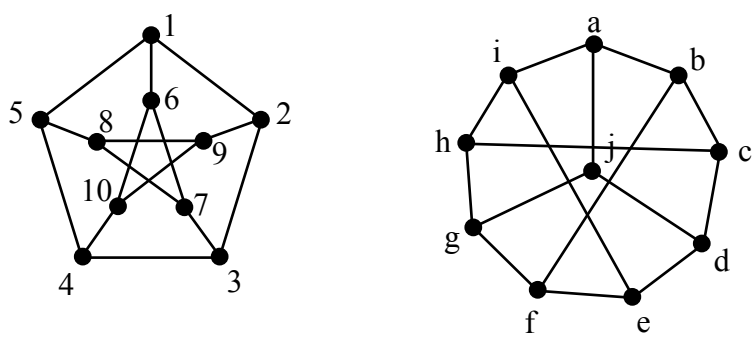


(c)



12. *Use planarity (including the theorems) to colour the planar graphs from question 1.

13. ***Prove the six-colour theorem.