Intermediate Math Circles
February 19, 2014
Contest Preparation III

Answers:

Problem Set 6:

1.	C	2.	A	3.	B	4.	[-6,6]	5.	$T = 3, U = 2$ and $T = 8, U = 6$
6.	69375	7.	C	8.	A	9.	C	10.	E

Australian Mathematics Competition - Intermediate 9:

1.	D	2.	E	3.	C	4.	D	5.	B
6.	D	7.	A	8.	A	9.	A	10.	D

Extra Problems:

1.	E	2.	C	3.	D

Solutions:

Problem Set 6:

1. When the sum of each pair is the same, the smallest number in the list must be grouped with the largest number in the list. The smallest number in the list is 9, the largest is 53. $9 + 53$ is equal to 62. Therefore, the value of every pair is 62. $62 - 15 = 47$, so the number paired with 15 is 47. The answer is C.

2. See past contest Pascal 2005, question 18.

3. First of all, there are 499 positive integers less than 500. Let's first count the numbers less than 500 that are divisible by 2. All even numbers under

500 are divisible by 2. There are $498 \div 2 = 249$ positive even numbers under 500. Next let's count the numbers less than 500 that are divisible by 3. The largest number smaller than 500 that is divisible by 3 is 498, where $498 \div 3 = 166$. Hence there are 166 positive integers under 500 that are divisible by 3. So far we have $499 - 249 - 166 = 85$ positive integers that are not divisible by 2 nor 3. But we've double counted the numbers that are both divisible by 2 and 3. By divisibility rules, we know that if a number is divisible by both 2 and 3, then it's divisible by 6. Therefore, let's add back the numbers that are divisible by 6. The largest number less than 500 divisible by 6 is 498, and $498 \div 6 = 83$. So, there are 83 numbers under 500 that are divisible by 6. Therefore there are a total of $499 - 249 - 166 + 83 = 167$ numbers under 500 that are not divisible by 2 or 3. The answer is B.

4. The largest possible interval for S occurs between $x - y$ is at a minimum and at a maximum. The minimum value of S occurs at $4 - 10 = -6$, the maximum value of S occurs at $12 - 6 = 6$. Hence the largest possible interval for S is $[-6, 6]$.

5. A number is divisible by 36 if and only if it's divisible by all possible combinations of its prime factors, namely $36 = 2 \times 2 \times 3 \times 3$. Since the divisibility rules for 4 and 9 are easily determined, we can use the fact that that a number is divisible by 36 if and only if it's divisible by both 4 and 9. Let's determine U first. A number is divisible by 4 if and only if its last 2 digits are divisible by 4. Namely, 72 and 76 are divisible by 4. We have 2 choices for U: $U = 2$ or $U = 6$.

   <u>Case 1:</u> Consider $U = 2$
   If $U = 2$ then a number is divisible by 9 if and only if the sum of its digits is divisible by 9. Then $9T672$ is divisible by 9 if and only if $T = 3$, as 27 is divisible by 9.

   <u>Case 2:</u> Consider $U = 6$
   If $U = 6$ then the only choice we have for T such that $9T676$ is divisible by 9 is $T = 8$, as 36 is divisible by 9.

   So the possible values for T and U are: $U = 2$, $T = 3$, and $U = 6$, $T = 8$

6. Let's give a name to these three-digit numbers for easy description, we'll call them "completely-odd". There are 5 odd one-digit numbers (namely 1,3,5,7, and 9). Hence there are 5 choices for the hundred's digit, 5 choices for the ten's digit, and 5 choices for the one's digit. We have a total of $5 \times 5 \times 5 = 125$ completely-odd numbers. Next, we must realize that adding three-digit numbers together is the same as adding the hundreds digits, multiplying by 100, adding the tens digits, multiplying by 10, and adding the ones digits, multiplying by 1 (For example, $531 + 317 = 100 \times (5+3) + 10 \times (3+1) + 1 \times (1+7) = 848$). The 1 in the hundred's digit occurs 25 times (since there are 25 out of the completely-odd numbers with a 1 in the hundred's digit), the 1 in the ten's digit occurs also 25 times as well (since it occurs 5 times in the 100's, 5 times in the 300's, etc.), and the 1 in the one's digit occurs 25 times as well. The same goes for the digits 3, 5, 7! , and 9. Hence the sum of all completely-odd numbers is:

$$\begin{aligned}
& 25 \times [100 \times (1+3+5+7+9) + 10 \times (1+3+5+7+9) \\
& +1 \times (1+3+5+7+9)] \\
= \; & [25 \times (1+3+5+7+9)] \times (100+10+1) \\
= \; & 625 \times 111 \\
= \; & 69375
\end{aligned}$$

7. See past contest Pascal 2005, question 20.

8. All even numbers are divisible by 2. We will expand 90! and try to divide it by 2 as many times as possible until the quotient is odd, then we know that we can no longer divide by 2. (Note, a number n divided by, say $8 = 2^3$, is the same as $n \div 2 \div 2 \div 2$). From 1 to 90, there are 45 even numbers. So all of these are divisible by 2. Then we can divide 90! by 2 at least 45 times. Performing this operation (dividing by 2 forty-five times) leaves a quotient of (45)(89)(44)(87)...(2)(3)(1)(1). Then we notice that every 4 of these numbers are still even. Namely, from 1 to 45, we have $44 \div 2 = 22$ even numbers left. So we can divide 2 into this quotient another 22 times. We get a new quotient of (45)(89)(22)(87)(43)(85)(21)...(1)(3)(1)(1). Now every 8 numbers is a even number. Namely from 1 to 22, there are $22 \div 2 = 11$ even numbers. So we can divide 2 into this new quotient another 11 times. Carrying on this pattern we get that there are $10 \div 2 = 5$ mor! e even

numbers that occur every 16 numbers; then there are $4 \div 2 = 2$ more even numbers that occur every 32 numbers, and finally $2 \div 2 = 1$ more even number that occur every 64 numbers.

In total, we may divide 2: $45 + 22 + 11 + 5 + 2 + 1 = 86$ times into 90!, i.e. 2^{86} divides 90!. Therefore, the highest power of 2 that divides evenly into 90! is 86. The answer is A.

9. See past contest Cayley 1997, question 22.

10. See past contest Cayley 1998, question 22.

Australian Mathematics Competition - Intermediate 9:

1.

$$\begin{aligned}(7a + 5b) - (5a - 7b) &= 7a + 5b - 5a + 7b \\ &= 2a + 12b\end{aligned}$$

The answer is (D).

2. The sum of angles along a straight line is 180°. So,

$$\begin{aligned}60^\circ + 20^\circ + 54^\circ + x^\circ + x^\circ &= 180^\circ \\ 134^\circ + 2x^\circ &= 180^\circ \\ 2x^\circ &= 180^\circ - 134^\circ \\ x^\circ &= \frac{46^\circ}{2} \\ x &= 23\end{aligned}$$

The answer is (E).

3.

$$\begin{aligned}\frac{\sqrt{20+x^2}}{\sqrt{20-x^2}} &= \frac{\sqrt{20+4^2}}{\sqrt{20-4^2}} \\ &= \frac{\sqrt{20+16}}{\sqrt{20-16}} \\ &= \frac{\sqrt{36}}{\sqrt{4}} \\ &= \frac{6}{2} \\ &= 3\end{aligned}$$

The answer is (C).

4. Dividing a number n by, say $8 = 2^3$, is the same as $n \div 2 \div 2 \div 2$. How many times can we divide 2 into 1000000?
We have:

$$\begin{aligned} 1000000 \div 2 &= 500000 && (1) \\ 500000 \div 2 &= 250000 && (2) \\ 250000 \div 2 &= 125000 && (3) \\ 125000 \div 2 &= 62500 && (4) \\ 62500 \div 2 &= 31250 && (5) \\ 31250 \div 2 &= 15625 \quad \text{which is an odd number} && (6) \end{aligned}$$

We've divided 2 into 1000000 six times. That is, 2^6 divides 1000000. The answer is (D).

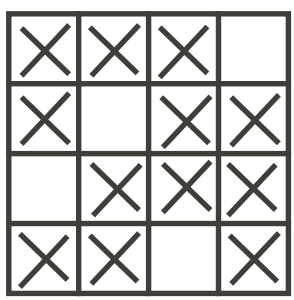
5. 1L of orange fruit juice is equivalent to 1000 mL of orange fruit juice. Since the 1000 mL contains 10% orange juice, there are $1000 \times 0.10 = 100$ mL of orange juice.
Let the amount of orange juice to be added be x, then we must have the original orange juice amount plus added in orange juice amount equal to half of total liquid in the fruit juice, where the total liquid in the fruit juice

is the original 1L plus the newly added in orange juice amount. Therefore,

$$\begin{aligned}
\frac{100+x}{1000+x} &= 0.5 \\
(100+x) &= 0.5 \times (1000+x) \\
100+x &= 500+0.5x \\
0.5x &= 400 \\
x &= 800
\end{aligned}$$

The amount of orange juice to be added is 800ml. The answer is (B).

6. One way to approach this problem is by using little strips of paper to represent the rows. We know that there are 4 ways to arrange 3 counters in four spots. If you put each of the ways on a strip of paper, you can rearrange the strips to satisfy the conditions. By doing this, we can see that you can arrange 12 counters in the boxes and satisfy all the conditions required. There are multiple solutions to this problem, but one of them is shown in the diagram below. It is not too difficult to show that 13 is not possible and that more than 10 is possible. You can reduce the answers to 11 and 12 without too much trouble. The answer is (D).



7. All positive integers less than 1000 have either 3 digits, 2 digits or 1 digit. The only 1 digit number whose sum of digits is 6 is 6 itself. For 2-digit numbers, there are 6 choices for the ten's digit, namely, $\{6, 5, 4, 3, 2, 1\}$. Each of these choices for the ten's digit has a unique corresponding one's digit. Therefore, there are 6 2-digit numbers whose sum of digits is 6. For 3-digit numbers, there are 6 choices for the hundred's digit. With each of these choices of the hundred's digit, the remaining 2 digits' sum needs to add up to 6 minus the value of hundred's digit. This gives us that the ten's and one's digit must add up to 0, 1, 2, 3, 4, or 5, depending on the value

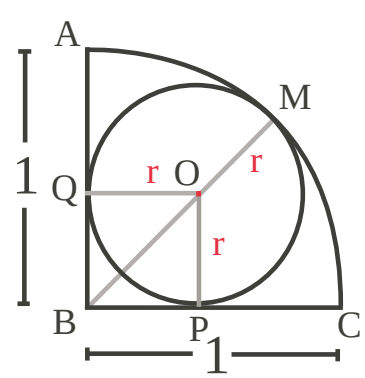
of the hundred's digit. Then, by the 2-digit number argument, there are respectively, 1, 2, 3, 4, 5, 6 choices for the ten's digit, and each of these choice of the ten's digit correspond to a unique one's digit. Therefore, there are a total of $1 + 2 + 3 + 4 + 5 + 6 = 21$ such 3-digit numbers.
Hence the number of positive integers less than 1000 whose sum of digits equals 6 is: $21 + 6 + 1 = 28$. The answer is (A).

8. Consider the ratio of John's money compared with Kevin and Robert's combined money. Before gambling, their ratio is $7 : 11$, after gambling, their ratio is $6 : 9$. Since $6 : 9 > 7 : 11$, John must've won the \$12, which came from the combined total of Kevin and Robert. Let $7x$ be the amount of money John had before, then $11x$ is the combined total of Kevin and Robert. Then,

$$\begin{aligned} \frac{7x+12}{11x-12} &= \frac{6}{9} \\ 66x - 72 &= 63x + 108 \\ 3x &= 180 \\ x &= 60 \end{aligned}$$

So John started with $7x = 7(60) =$\$420 dollars. The answer is (A).

9. Fact: The largest circle that can be drawn in this quarter circle must be an inscribed circle, i.e. it touches the quarter circle on the arc and the 2 radii. Consider the diagram below.



From the diagram, we see that $OQ \perp AB$ and $OP \perp BC$ (line from center of circle to a tangent always intersect the tangent at $90°$). Then $\triangle OBP = \triangle OBQ$ are two identical right-angled triangles, with $OP = OQ = BP =$

$BQ = r$. Then OB has length, by the Pythagorean Theorem, $\sqrt{r^2 + r^2} = \sqrt{2}r$. Also note that BM is also a radius of the bigger quarter circle ABC. So $BM = BA = BC = 1$. But $BM = OB + OM = \sqrt{2}r + r$. Therefore,

$$\begin{aligned}
\sqrt{2}r + r &= 1 \\
r(\sqrt{2}+1) &= 1 \\
r &= \frac{1}{\sqrt{2}+1} \\
&= \frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} \\
&= \sqrt{2}-1
\end{aligned}$$

The answer is (A).

10. First, we will convert the fractions $\frac{7}{10}$ and $\frac{11}{15}$ into fractions with the same denominator. Namely,

$$\frac{21}{30} < \frac{p}{q} < \frac{22}{30}$$

Of course, there are many fractions that lie between $\frac{21}{30}$ and $\frac{22}{30}$, but we are looking for the fraction with minimal denominator. One way to solve this is to multiply through by $30q$ to get an equivalency inequality

$$21q < 30p < 22q.$$

The question now becomes:

Which is the least positive integer q such that there is a multiple of 30 lying between $21q$ and $22q$? Let's simply test the 5 answers, starting with the smallest number.

If $q = 6$ then we are looking for a multiple of 30 between 126 and 132 which does not exist.

If $q = 7$ then we want an multiple of 30 between 147 and 154. We have $150 = 30 \times 5$. So the answer is (D).

Note that we actually found more than we needed. We found the actual fraction $\frac{p}{q} = \frac{5}{7}$.

To complete the picture (just for fun), we can convince ourselves that there are indeed no multiples of 30 between

(q=1)	21	and	22
(q=2)	42	and	44
(q=3)	63	and	66
(q=4)	84	and	88
(q=5)	105	and	110

Extra Problems:

1. Let's let $n = 1996^{1996}$. Factoring 1996, we get $1996 = 2 \times 2 \times 499$ and so

$$n = 1996^{1996} = (2^2 \cdot 499)^{1996} = 2^{2\times 1996} \cdot 499^{1996}.$$

So all of the divisors of n are of the form $2^p \cdot 499^q$ for some $0 \leq p \leq (2)(1996)$ and $0 \leq q \leq 1996$. For a divisor to be a perfect square, it must have an *even* number of factors of both 2 and 499. There are

$$\frac{(2)(1996)}{2} + 1 = 1997$$

even numbers less than or equal to $2 \cdot 1996$ (including 0), and there are

$$\frac{(1996)}{2} + 1 = 998 + 1 = 999$$

even numbers less than or equal to 1996 (including 0). So there are 1997 choices for p and 999 choices for q, totalling $k = 1997 \times 999 = 1,995,003$ divisors that are perfect squares. The sum of the digits of k is $1 + 9 + 9 + 5 + 0 + 0 + 3 = 27$ so the answer is (E).

2. See past contest 2002 American Mathematics Competition 10A, Question 25.

3. See past contest Cayley 1998, Question 25.