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Enriching Mathematics and Computer Science for 50 years

Senior Math Circles

October 30, 2013

Surfaces - Problem Set 1 Solutions

Here are solutions to some of the questions. Not all solutions appear here for two reasons. First, a whole bunch of the questions are answered (at least partially) somewhere in the notes, usually after the question has been posed. Second, I don't want to ruin the fun for you by giving you all the answers!

2c. $\chi(PP) = 1$.

2d. $\chi(\text{2-holed torus}) = -2$.

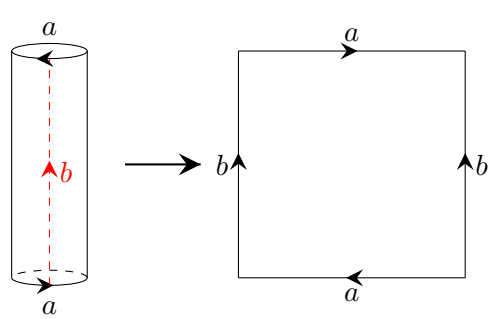
4. Adding a handle turns out to be the same as taking the connected sum with a torus. We see later on in the notes that $\chi(A\#B) = \chi(A) + \chi(B) - 2$ (this is Theorem 2 in the notes) and since $\chi(T) = 0$ we see that adding a handle decreases the Euler characteristic of any surface by 2.

7a. A projective plane.

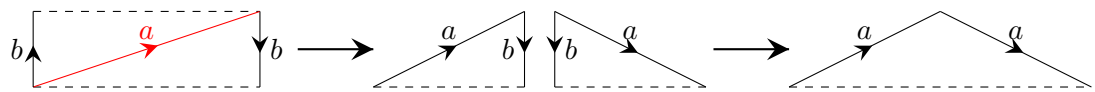
7b. The connected sum of two projective planes (or a Klein bottle, see the end of the notes for why these are just two different names for the same surface).

7c. Adding a crosscap turns out to be the same as taking the connected sum with a projective plane. So, by the same reasoning in question 4 above, we see that adding a crosscap decreases the Euler characteristic of any surface by 1.

8. *Klein Bottle:*
For the Klein bottle, we simply take the cylindrical model we had in the definition of the surface, and cut along its length as showed below as the red dashed line. We then open up the cylinder (keeping track of what's being glued to what) and voila, we have our planar model.



Projective Plane:
For the projective plane, we first need to view the Möbius strip a little differently. We can cut and paste the Möbius strip as below to make it more obvious that it has only one edge. In the model below, the dashed line represents the edge of the Möbius strip and the red line is where the cut is being made.



Now that we have this, we can glue the one edge (below labelled b) to the edge of the disk, and we end up with a planar model of the projective plane!