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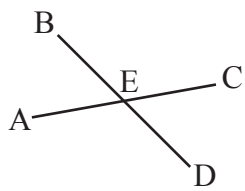
Grade 7/8 Math Circles

November 5/6/7, 2013

Congruence and Similarity - Solutions

Here are a couple of results that may be helpful in the completion of this problem set:



Opposite angles

Angles that are opposite each other when two lines cross are always equal in measure.

In the diagram above, $\angle BEC = \angle AED$ and $\angle BEA = \angle CED$

Angles along a straight line

Two or more adjacent angles that lie along a straight line must add up to $180°$.

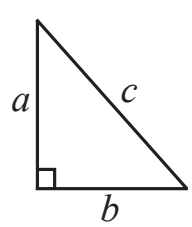
In the diagram above, $\angle BEC$ and $\angle CED$ are adjacent and lie along the straight line segment BD. Therefore, $\angle BEC + \angle CED = 180°$

See if you can spot the three other pairs of angles with the same property.

Pythagorean Theorem

Given a **right-angled** triangle with hypotenuse of length c and legs of length a and b, it is true that

$$\boxed{c^2 = a^2 + b^2}$$



Note: The strict format of the proofs in these solutions is used to make the solutions clear. Proofs can be presented in different formats as long as each statement is given by clear reasoning.

1. (a) (1) $\triangle ABC$ and $\triangle DEF$ are equilateral (given)

   (2) $|AB| = |BC| = |CA|$ and $|DE| = |EF| = |FD|$ (properties of equilateral triangles)

   (3) Hence, $\dfrac{|AB|}{|DE|} = \dfrac{|BC|}{|EF|} = \dfrac{|CA|}{|FD|}$

   Therefore, $\triangle ABC \sim \triangle DEF$ (SSS)

   $\square$

   (b) (1) Assume that $\triangle ABC \sim \triangle DEF$

   (2) Then, all corresponding angles of $\triangle ABC$ and $\triangle DEF$ are equal in measure (properties of similarity)

   (3) But $\triangle ABC$ is obtuse and $\triangle DEF$ is acute (given)

   (4) Then, $\triangle ABC$ has one angle that is greater than $90°$ (by definition of obtuse)

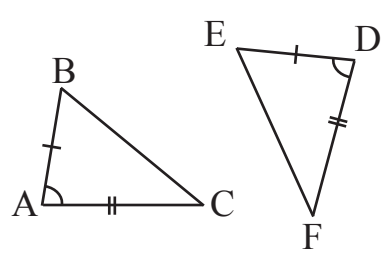
   (5) And $\triangle DEF$ has no angles greater than $90°$ (by definition of acute)

   (6) Thus, $\triangle ABC$ and $\triangle DEF$ have at least one pair of corresponding angles that are not equal in measure (follows from (4) and (5))

   Statement (2) contradicts statement (6), and so the assumption that $\triangle ABC \sim \triangle DEF$ must be incorrect. That is, it must be true that $\triangle ABC \nsim \triangle DEF$

   $\square$

   (c) We can draw a picture to help visualize the situation:



   It is clear from the diagram, that we have the SAS case. But to prove that $\triangle ABC \sim \triangle DEF$, we should write out our thought process.

   (1) $|AB| = |DE|$ and so $\dfrac{|AB|}{|DE|} = 1$ (given/diagram)

   (2) $|AC| = |DF|$ and so $\dfrac{|AC|}{|DF|} = 1$ (given/diagram)

③ $\angle BAC = \angle EDF$ and the angles are contained (given/diagram)

Therefore, $\triangle ABC \sim \triangle DEF$ (SAS)

$\square$

(d) ① AB and DE, BC and EF, and AC and DF are the three pairs of corresponding sides of the triangles $\triangle ABC$ and $\triangle DEF$ (match vertices)

② $\dfrac{|DE|}{|AB|} = \dfrac{|EF|}{|BC|} = \dfrac{|DF|}{|AC|} = 2$ (given)

Therefore, $\triangle ABC \sim \triangle DEF$ (SSS)

$\square$

2. (a) *Step 1: Prove that that the triangles are similar:*

① $\dfrac{|AB|}{|DE|} = \dfrac{5\text{ cm}}{5\text{ cm}} = 1$ (given)

② $\dfrac{|BC|}{|EF|} = \dfrac{4\text{ cm}}{4\text{ cm}} = 1$ (given)

③ $\dfrac{|CA|}{|FD|} = \dfrac{4\text{ cm}}{40\text{ mm}} = \dfrac{4\text{ cm}}{4\text{ cm}} = 1$ (given)

Therefore, $\triangle ABC \sim \triangle DEF$ (SSS)

Step 2: Prove that the triangles are congruent:

① $\triangle ABC \sim \triangle DEF$ (Step 1)

② $|AB| = |DE|$ and AB and DE are corresponding sides of the triangles $\triangle ABC$ and $\triangle DEF$ (given, match vertices)

Therefore, $\triangle ABC \cong \triangle DEF$ (congruence test)

$\square$

(b) *Step 1: Show that the triangles are similar*

① $|AB| = |DE|$ and so $\dfrac{|AB|}{|DE|} = 1$ (given)

② $|AC| = |DF|$ and so $\dfrac{|AC|}{|DF|} = 1$ (given)

③ $\angle ABC = \angle DEF = 90^\circ$ (given)

Therefore, $\triangle ABC \sim \triangle DEF$ (SSRA)

Step 2: Prove that the triangles are congruent:

① $\triangle ABC \sim \triangle DEF$ (Step 1)

② $|AB| = |DE|$ and AB and DE are corresponding sides of the triangles $\triangle ABC$ and $\triangle DEF$ (given, match vertices)

Therefore, $\triangle ABC \cong \triangle DEF$ (congruence test)

$\square$

(c) (1) Assume that $\triangle ABC \cong \triangle DEF$

(2) Then, all the corresponding sides of the triangles are equal in length. That is,

$$\frac{|AB|}{|DE|} = \frac{|BC|}{|EF|} = \frac{|AC|}{|DF|} = 1$$

(given, match vertices)

(3) But, $\dfrac{|AC|}{|DF|} = \dfrac{5 \text{ m}}{10 \text{ m}} = \dfrac{1}{2} \neq 1$ (given)

Statement (2) contradicts statement (3), and so the assumption that the triangles are congruent must be false. That is, it must be true that $\triangle ABC \ncong \triangle DEF$

$\square$

(d) *Step 1: Show that the triangles are similar*

(1) $\angle CAB = \angle FDE = 45°$ (given)

(2) $\angle BCA = \angle EFD = 45°$ (given)

Therefore, $\triangle ABC \sim \triangle DEF$ (AA)

Step 2: Prove that the triangles are congruent:

(1) $\triangle ABC \sim \triangle DEF$ (Step 1)

(2) $|CA| = 102 \text{ mm} = |FD| = 0.102 \text{ m}$ and CA and FD are corresponding sides of the triangles $\triangle ABC$ and $\triangle DEF$ (given, match vertices)

Therefore, $\triangle ABC \cong \triangle DEF$ (congruence test)

$\square$

Also, the triangles are right-angled isosceles triangles.

3. A regular polygon has all sides of equal length and all angles of equal measure. (by definition)

   By dividing regular polygons into triangles, it can be shown that the sum of all their interior angles is a fixed number.

   Therefore, since all the interior angles in a regular polygon are of equal measure, then we have that the measure of each angle is given by

$$\frac{\text{the sum of the interior angles}}{\text{the number of interior angles}}$$

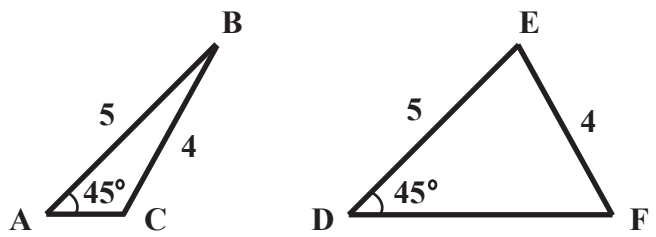
   So any two regular polygons with the same number of sides will have all of their interior angles equal. For example, all equilateral triangles have interior angles of 60°, all squares have interior angles of 90°, regular pentagons: 108°, regular hexagons: 120°.

Thus, any two regular polygons with the same number of sides must have the same shape, even if they are different sizes.

Therefore, any two regular polygons with the same number of sides are just scaled copies of each other and so they are similar.

□

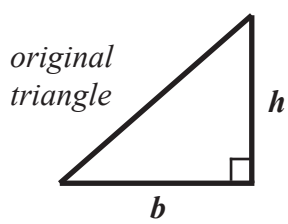
4. The triangles $\triangle ABC$ and $\triangle DEF$ can be drawn as below.



These triangles are not congruent even though they both have two pairs of corresponding sides equal and one pair of non-contained angles equal. This is clear because one triangle cannot be transformed (ie. translated, rotated, reflected) onto the other triangle.

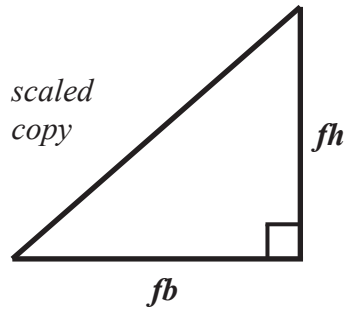
This is why SSA does not provide enough information to be a similarity or congruence test.

5. (a) Let b represent the length of the base of the original triangle, and let h represent the height of the original triangle.



Then the area of the original triangle is $\frac{1}{2}bh$ (formula for area of triangle)
Since the original triangle is right-angled, the base and height are simply the legs (sides adjacent to the right angle) of the triangle.

When the original triangle is scaled by a factor of f, the lengths of its edges are multiplied by f. That is, the scaled copy of the triangle has height fh and base fb.

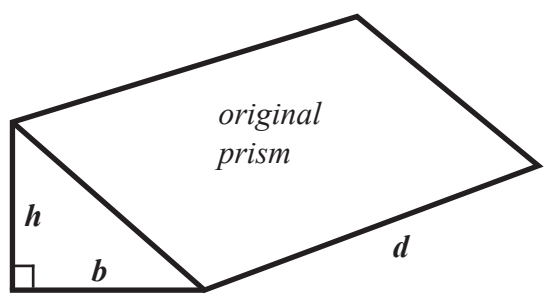


Therefore, the area of the scaled triangle has area

$$\frac{1}{2}(fb)(fh) = (f^2)\left(\frac{1}{2}bh\right) = (f^2)(\text{area of original triangle})$$

Thus, the area of a right-angled triangle is scaled by a factor of f^2 when its edges are scaled by a factor of f.

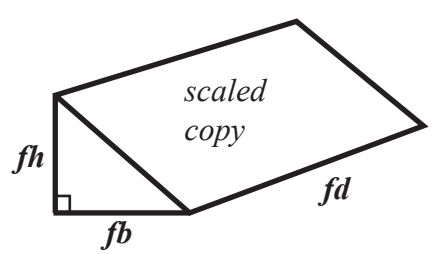
(b) Let b represent the length of the base of the triangular face of the original prism, and let h represent the height of the triangular face of the original prism. Finally, let d represent the depth of the prism.



The volume of the original prism is $\frac{1}{2}bhd$ (formula for volume of a triangular prism)

Since the triangular face of the original prism is right-angled, the base and height are simply the legs of the triangle.

When the original prism is scaled by a factor of f, the lengths of its edges are multiplied by f. That is, the scaled copy of the triangular prism looks as shown below:



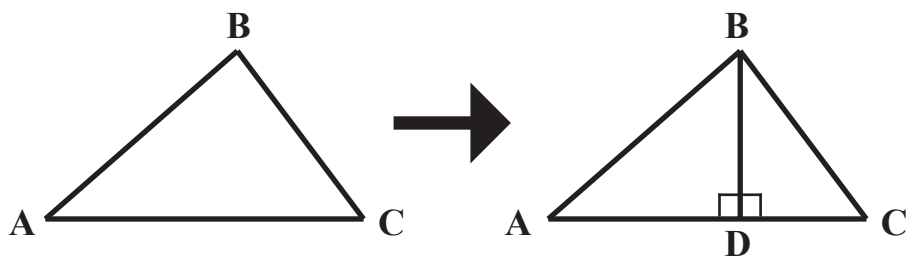
Therefore, the volume of the scaled prism is

$$\frac{1}{2}(fb)(fh)(fd) = (f^3)\left(\frac{1}{2}bhd\right) = (f^3)(\text{volume of original prism})$$

Thus, the volume of a right triangular prism is scaled by a factor of f^3 when the edges of the prism are scaled by a factor of f.

(c) *We can show that the results hold by breaking up the larger shapes and prisms into triangles and triangular prisms, and then further into right-angled triangles and right triangular prisms.*

We can divide any triangle into two right-angled triangles as seen below:



Suppose a triangle, say $\triangle ABC$, is scaled by a factor f. Let's call the scaled copy $\triangle A'B'C'$.

We have that $\triangle ABC \sim \triangle A'B'C'$ by SSS or by the definition of similarity. Therefore, all the corresponding angles of the triangles $\triangle ABC$ and $\triangle A'B'C'$ are equal in measure.

So if we create a point D' on the line segment $A'C'$ so that $B'D'$ is perpendicular to $A'C'$ (just as BD is perpendicular to AC), then we will have two pairs of equal corresponding angles between $\triangle ABD$ and $\triangle A'B'D'$:

1 $\angle BAD = \angle B'A'D'$

2 $\angle ADB = \angle A'D'B' = 90°$

Therefore, $\triangle ABD \sim \triangle A'B'D'$ (AA)

And since $\dfrac{|A'B'|}{|AB|} = f$, then by properties of similarity, we must have that

$$\frac{|B'D'|}{|BD|} = \frac{|A'D'|}{|AD|} = f$$

Similarly, we can show that $\triangle BCD \sim \triangle B'C'D'$ and that

$$\frac{|D'C'|}{|DC|} = \frac{|B'C'|}{|BC|} = f$$

Thus, we can see that scaling any triangle by a factor by a factor f is basically the same thing as scaling the two right triangles that we can divide it into.

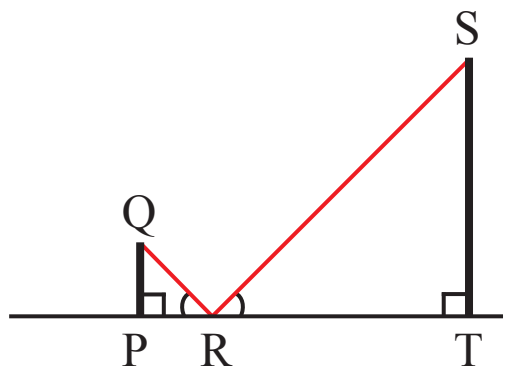
By the result of part (a), we know that the areas of both of the right triangles $\triangle ABD$ and $\triangle BCD$ scale by a factor of f^2. Since the sum of the areas of $\triangle ABD$ and $\triangle BCD$ is the area of $\triangle ABC$, then the area of $\triangle ABC$ will also scale by a factor of f^2 when its edges are scaled by a factor of f.

Similarly, we can divide any triangular prism up into two right triangular prisms and find that the volume of any triangular prism scales by a factor of f^3 when its edges are scaled by a factor of f.

Finally, we can divide any polygon into a collection of triangles. In a similar method as above, we can show that the areas of these triangles scale by a factor of f^2 when the edges of the polygon are scaled by a factor of f. Therefore, since the area of the polygon is the sum of the areas of these triangles, the area of the polygon scales by a factor of f^2 when its edges are scaled by a factor of f.

Any prism with a polygonal base can be broken up into a collection of triangular prisms. It can be shown that the volumes of these prisms scale by a factor of f^3 when the edges of the polygonal-base prism are scaled by a factor of f. Since the volume of the polygonal-base prism is the sum of the volumes of these triangular prisms, then we have that the volume of any polygonal-base prism scales by a factor of f^3 when its edges are scaled by a factor of f.

6. The first thing we do is draw a picture (not to scale):



In the picture, the line segment PQ is you (specifically your eyes to the ground), the line segment TS is the flagpole, and the mirror is positioned at point R. The light ray from the top of the flagpole to your eyes travels the path in red.

We know that $\angle QRP = \angle SRT$ because the reflected light leaves the mirror at the same angle from the mirror as it approached the mirror. This is a general law of reflection.

We also know that $\angle QPR = \angle STR = 90°$ (from the hint, or a good assumption)

Therefore, $\triangle PQR \sim \triangle TSR$ (AA)

Since the triangles are similar, by properties of similarity, all of the corresponding sides of the triangles must be in proportion. That is, we must have that

$$\frac{|TS|}{|PQ|} = \frac{|TR|}{|PR|} = \frac{|SR|}{|QR|}$$

From the information given in the problem, we know that $|PQ| = 5$, $|TR| = 7$, and $|PR| = 2$ (all in units of feet). Thus, we can solve for the height of the flagpole, $|TS|$, as follows:

$$\begin{aligned}
\frac{|TS|}{5} &= \frac{7}{2} \\
|TS| &= \left(\frac{7}{2}\right)(5) \\
|TS| &= \frac{35}{2} \\
|TS| &= 17.5
\end{aligned}$$

Since all the lengths were measured in units of feet, the flagpole is 17.5 feet tall.

7. (1) $\angle QRP = \angle TRS$ (diagram, opposite angles)

(2) $\angle PQR = \angle STR$ (diagram)

Therefore, $\triangle PQR \sim \triangle STR$ (AA)

Thus, by properties of similar triangles,

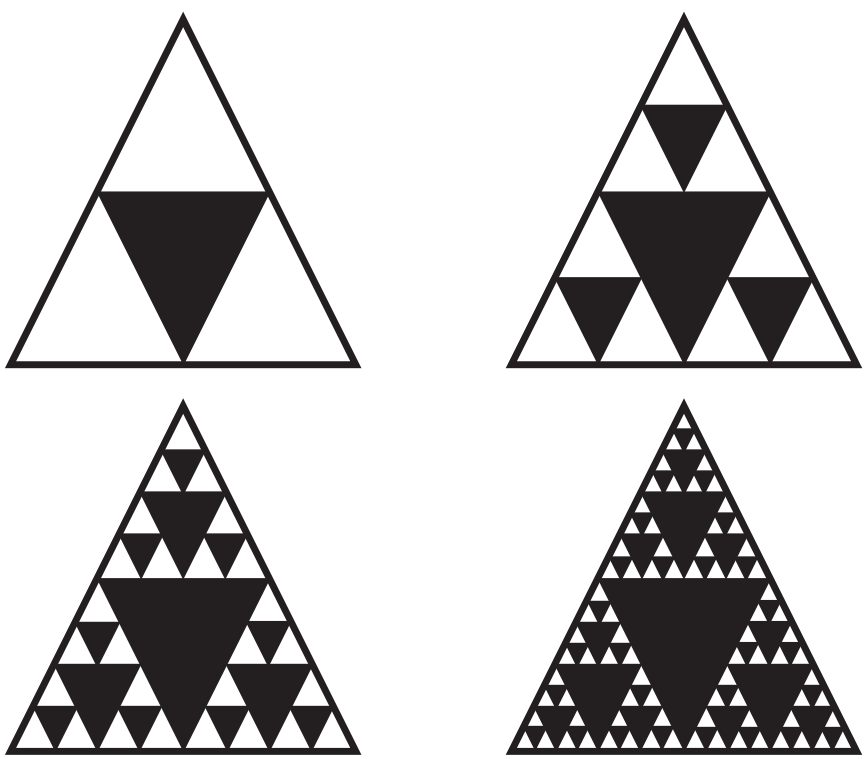
$$\frac{|QP|}{|TS|} = \frac{|QR|}{|TR|}$$

Using the information given, we have that

$$\begin{aligned}
\frac{|QP|}{9} &= \frac{7}{10.5} \\
\frac{|QP|}{9} &= \frac{2}{3} \\
|QP| &= \left(\frac{2}{3}\right)(9) \\
|QP| &= 6
\end{aligned}$$

Hence, the Grand River is 6 m wide between points Q and P.

8. (a) Here is what the first four iterations look like with the specified shading pattern:



(b) The triangles created in any given iteration are scaled by a factor of $\frac{1}{2}$ to create the triangles of the next iteration.

(c) From question 5 (c), we know that the area of the triangles scales by $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

(d) As seen in the images above, 3 unshaded smaller triangles are made from every one unshaded triangle in any given iteration.

(e) In any given iteration, there are 3 times as many unshaded triangles than in the previous iteration, and the new unshaded triangles have one quarter of the area of the unshaded triangles from the previous iteration.

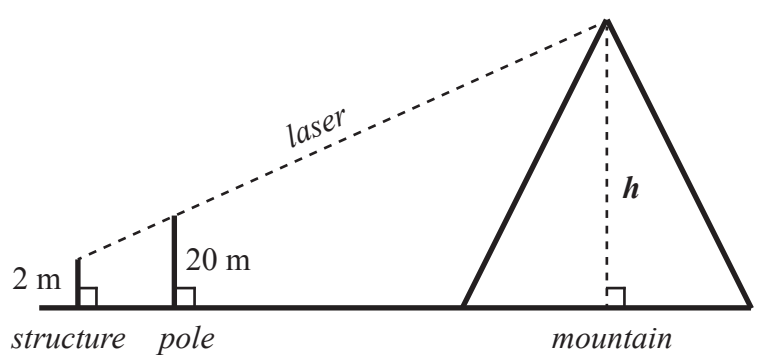
Therefore, the fraction of the area of the original triangle that is made up of unshaded triangles after the nth iteration is

$$(3)^n \left(\frac{1}{4}\right)^n = \left(\frac{3}{4}\right)^n$$

You can also think of it as each unshaded triangle losing $\frac{1}{4}$ of its area in every iteration.

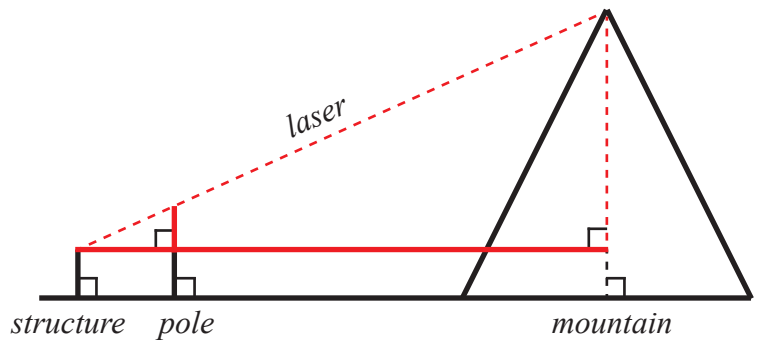
(f) Once again, we encounter the idea of a limit. As we perform an infinite number of iterations, the area of the unshaded triangles approaches zero!

9. *Step 1: Draw a picture! Note that the diagram below is not to scale.*

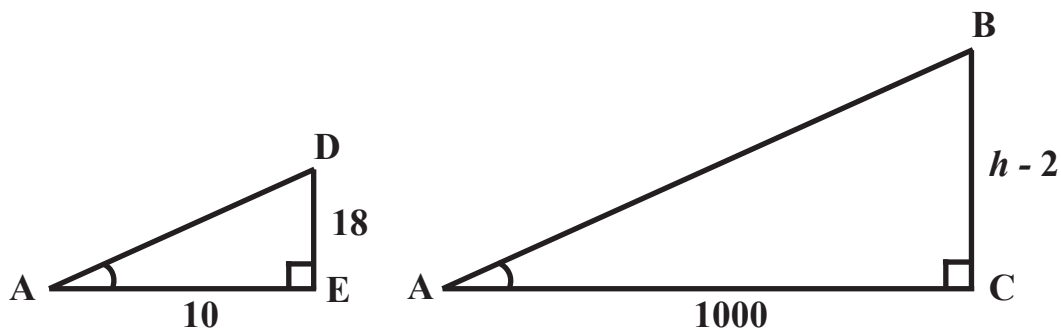


Our goal is to find the height, h, of the mountain.

Notice that this problem is very similar to the opening problem of the lesson. In fact, if we create two triangles as seen below (in red), the problem becomes nearly identical.



These triangles can be separated and labelled as seen here.



Step 2: Show that the triangles are similar:

1. $\angle DAE = \angle BAC$ (clear from the original diagram)
2. $\angle AED = \angle ACB$ (clear from the original diagram)

Therefore, $\triangle ADE \sim \triangle ABC$ (AA)

Step 3: Use properties of similarity to solve for h:

1. $\triangle ADE \sim \triangle ABC$ (Step 2)
2. $\dfrac{|AC|}{|AE|} = \dfrac{1000}{10} = 100$ (diagram)
3. DE and BC are corresponding sides of $\triangle ADE$ and $\triangle ABC$ (diagram)

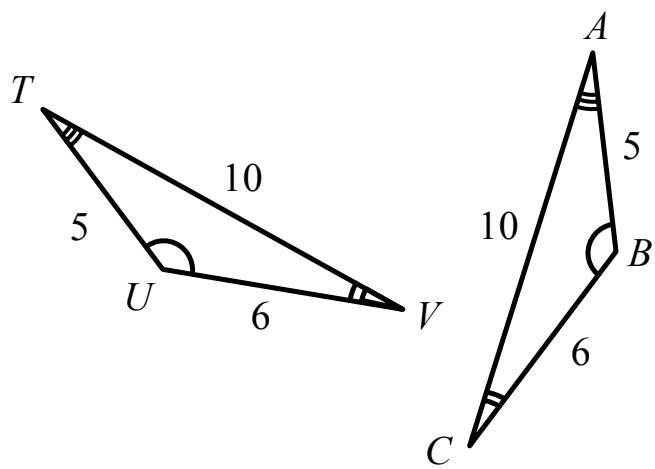
Therefore, $\dfrac{|BD|}{|DE|} = 100$

Thus, we substitute and solve for h,

$$\begin{aligned}
\frac{h-2}{18} &= 100 \\
h-2 &= (100)(18) \\
h-2 &= 1800 \\
h &= 1800+2 \\
h &= 1802
\end{aligned}$$

Since all lengths were in units of metres, the height of the mountain is solved to be 1802 metres.

10. If $\triangle ABC \cong \triangle TUV$ then all the corresponding sides must have equal length and all the corresponding angles must be equal. Thus, we can fill in the diagram as seen below:



11. *Step 1: Show that the triangles $\triangle SPQ$ and $\triangle QRS$ are similar:*

    (1) $\angle SPQ = \angle QRS$ (diagram)

    (2) $\angle PSQ = \angle RQS$ (diagram)

    Therefore, $\triangle SPQ \sim \triangle QRS$ (AA)

    Step 2: Show that the triangles $\triangle SPQ$ and $\triangle QRS$ are congruent:

    (1) $\triangle SPQ \sim \triangle QRS$ (Step 1)

    (2) $|SQ| = |QS|$, SQ and QS are corresponding sides of $\triangle SPQ$ and $\triangle QRS$ (clearly/diagram)

    Therefore, $\triangle SPQ \cong \triangle QRS$ (congruence test)

Step 3: Use properties of congruence:

1 $\triangle SPQ \cong \triangle QRS$ (Step 2)

2 PQ and RS are corresponding sides of $\triangle SPQ$ and $\triangle QRS$ (digram)

Therefore, $|PQ| = |RS|$ (properties of congruence)

$\square$

12. *Step 1: Show that the triangles $\triangle ABC$ and $\triangle ADC$ are similar:*

1 $|BC| = |CD| = |DC|$ or $\dfrac{|BC|}{|DC|} = 1$ (given)

2 $\dfrac{|CA|}{|CA|} = 1$ (clearly/diagram)

3 $\angle BCA = \angle DCA$ and the angles are contained (given/diagram)

Therefore, $\triangle ABC \sim \triangle ADC$ (SAS)

Step 2: Show that the triangles $\triangle ABC$ and $\triangle ADC$ are congruent:

1 $\triangle ABC \sim \triangle ADC$ (Step 1)

2 $|BC| = |CD| = |DC|$, BC and DC are corresponding sides of $\triangle ABC$ and $\triangle ADC$ (given/diagram)

Therefore, $\triangle ABC \cong \triangle ADC$ (congruence test)

$\square$

13. *Step 1: Show that $\angle XWY = 90°$:*

1 $\angle XWY + \angle XWZ = 180°$ (diagram, angles along a straight line)

2 $\angle XWZ = 90°$ (diagram)

Therefore, $\angle XWY + 90° = 180°$, and so $\angle XWY = 90°$ (substitution/algebra)

Step 2: Show that the triangles $\triangle XYW$ and $\triangle XZW$ are similar:

1 $\angle XWY = \angle XWZ = 90°$ (Step 1)

2 $|XY| = |XZ|$ or $\dfrac{|XY|}{|XZ|} = 1$ (digram)

3 $|XW| = |XW|$ or $\dfrac{|XW|}{|XW|} = 1$ (clearly/diagram)

Therefore, $\triangle XYW \sim \triangle XZW$ (SSRA)

Step 3: Use properties of similarity:

1 $\triangle XYW \sim \triangle XZW$ (Step 2)

2 $\angle XYW = \angle XYZ$ and $\angle XZW = \angle XZY$ are corresponding angles of triangles $\triangle XYW$ and $\triangle XZW$ (diagram)

Therefore, $\angle XZY = \angle XYZ$ (properties of similarity)

$\square$

14. *Step 1: Show that the triangles $\triangle ABC$ and $\triangle EBD$ are similar:*

1 $\angle BAC = \angle BED$ (given)

2 $\angle ABC = \angle EBD$ (clearly/diagram)

Therefore, $\triangle ABC \sim \triangle EBD$ (AA)

Step 2: Show that the triangles $\triangle ABC$ and $\triangle EBD$ are congruent:

1 $\triangle ABC \sim \triangle EBD$ (Step 1)

2 $|AB| = |BE| = |EB|$, AB and EB are corresponding sides of $\triangle ABC$ and $\triangle EBD$ (given/diagram)

Therefore, $\triangle ABC \cong \triangle EBD$ (congruence test)

Step 3: Use properties of congruence:

1 $\triangle ABC \cong \triangle EBD$ (Step 2)

2 CB and DB are corresponding sides of triangles $\triangle ABC$ and $\triangle EBD$ (diagram)

Therefore, $|CB| = |DB|$ (properties of congruence)

Step 4: Do some thinking:

1 $|AD| + |DB| = |AB|$ (diagram)

2 $|EC| + |CB| = |EB|$ (diagram)

3 $|AB| = |BE| = |EB|$ (given)

4 $|CB| = |DB|$ (Step 3)

Therefore,

$$\begin{aligned} |AD| + |DB| &= |EC| + |CB| \\ |AD| + |DB| &= |EC| + |DB| \\ |AD| &= |EC| \end{aligned}$$

$\square$

15. *Step 1: Show that $\angle AEC = \angle BEC$:*

(1) $\angle AED + \angle AEC = 180°$ (diagram, angles along a straight line)

(2) $\angle BED + \angle BEC = 180°$ (diagram, angles along a straight line)

(3) $\angle AED = \angle BED$ (given)

Therefore,

$$\begin{aligned} \angle AED + \angle AEC &= \angle BED + \angle BEC \\ \angle BED + \angle AEC &= \angle BED + \angle BEC \\ \angle AEC &= \angle BEC \end{aligned}$$

Step 2: Show that the triangles $\triangle ACE$ and $\triangle BCE$ are similar:

(1) $\angle AEC = \angle BEC$ (Step 1)

(2) $\angle ACD = \angle ACE = \angle BCD = \angle BCE$ (given, diagram)

Therefore, $\triangle ACE \sim \triangle BCE$ (AA)

Step 3: Show that the triangles $\triangle ACE$ and $\triangle BCE$ are congruent:

(1) $\triangle ACE \sim \triangle BCE$ (Step 2)

(2) $|CE| = |CE|$, CE and CE are corresponding sides of $\triangle ACE$ and $\triangle BCE$ (clearly/diagram)

Therefore, $\triangle ACE \cong \triangle BCE$ (congruence test)

Step 4: Use properties of congruence:

(1) $\triangle ACE \cong \triangle BCE$ (Step 3)

(2) AE and BE are corresponding sides of $\triangle ACE$ and $\triangle BCE$ (diagram)

Therefore, $|AE| = |BE|$ (properties of congruence)

Step 5: Show that the triangles $\triangle AED$ and $\triangle BED$ are similar:

(1) $|AE| = |BE|$ or $\dfrac{|AE|}{|BE|} = 1$ (Step 4)

(2) $|ED| = |ED|$ or $\dfrac{|ED|}{|ED|} = 1$ (clearly/diagram)

(3) $\angle AED = \angle BED$ and the angles are contained (given/diagram)

Therefore, $\triangle AED \sim \triangle BED$ (SAS)

Step 6: Use properties of similarity:

(1) $\triangle AED \sim \triangle BED$ (Step 5)

(2) $\angle EAD$ and $\angle EBD$ are corresponding angles of $\triangle AED$ and $\triangle BED$ (diagram)

Therefore, $\angle EAD = \angle EBD$ (properties of similarity)

$\square$

16. *Step 1: Show that the triangles $\triangle ABM$ and $\triangle DCM$ are similar:*

(1) $\angle ABM = \angle DCM$ (diagram)

(2) $\angle BMA = \angle CMD$ (diagram, opposite angles)

Therefore, $\triangle ABM \sim \triangle DCM$ (AA)

Step 2: Use properties of similarity:

(1) $\triangle ABM \sim \triangle DCM$ (Step 1)

(2) AM and DM are corresponding sides of triangles $\triangle ABM$ and $\triangle DCM$ (diagram)

(3) BM and CM are corresponding sides of triangles $\triangle ABM$ and $\triangle DCM$ (diagram)

(4) $\dfrac{|AB|}{|DC|} = \dfrac{6}{3} = 2$ (diagram)

Therefore, by properties of similarity,

$$\frac{|AM|}{|DM|} = \frac{|BM|}{|CM|} = 2$$

Step 3: Get creative, find $|AM|$ and $|DM|$:

(1) Let $|AM| = x$.

(2) $|AD| = |AM| + |MD| = |AM| + |DM| = 15$ (given/diagram)

Therefore, $x + |DM| = 15$ or $|DM| = 15 - x$ (substitution/algebra)

(3) $\dfrac{|AM|}{|DM|} = 2$ (Step 2)

Therefore, by substitution,

$$\begin{aligned}
\frac{x}{15-x} &= 2 \\
x &= (2)(15-x) \\
x &= 30 - 2x \\
x + 2x &= 30 \\
3x &= 30 \\
x &= \frac{30}{3} \\
x &= 10
\end{aligned}$$

Thus, $|AM| = x = 10$ and $|DM| = 15 - x = 15 - 10 = 5$

Step 4: Use Pythagorean Theorem to find $|BM|$:

1 The triangle $\triangle ABM$ is right-angled (diagram)

2 $|AB| = 6$ (diagram)

3 $|AM| = 10$ (Step 3)

Therefore, by Pythagorean Theorem,

$$\begin{aligned} 6^2 + |BM|^2 &= 10^2 \\ 36 + |BM|^2 &= 100 \\ |BM|^2 &= 100 - 36 \\ |BM|^2 &= 64 \\ |BM| &= \sqrt{64} \\ |BM| &= 8 \end{aligned}$$

Step 5: Use properties of similarity to find $|CM|$:

1 $\dfrac{|BM|}{|CM|} = 2$ (Step 2)

2 $|BM| = 8$ (Step 5)

Therefore, $\dfrac{8}{|CM|} = 2$, and so $|CM| = 4$

Step 6: Calculate the perimeter:

From our work so far, we have determined that $\triangle ABM$ has sides of length 6, 8, and 10, and $\triangle DCM$ has sides of length 3, 4, 5.

Therefore, the perimeter of $\triangle ABM$ is $6 + 8 + 10 = 24$ units, and the perimeter of $\triangle DCM$ is $3 + 4 + 5 = 12$ units.

17. **Part 1:** Show that $\angle CBE + \angle CDE = 180°$

Step 1: Show that $\triangle ADE$ is similar to $\triangle ABC$:

1 $|AD| \times |AC| = |AE| \times |AB|$ or $\dfrac{|AD|}{|AB|} = \dfrac{|AE|}{|AC|}$ (given, algebra)

2 $\angle BAC = \angle EAD$ and the angles are contained (clearly/diagram)

Therefore, $\triangle ADE \sim \triangle ABC$ (SAS)

Step 2: Use properties of similarity:

1 $\triangle ADE \sim \triangle ABC$ (Step 1)

2 $\angle ADE$ and $\angle ABC$ are corresponding angles of $\triangle ADE$ and $\triangle ABC$ (diagram / match vertices)

Therefore, $\angle ADE = \angle ABC = \angle CBE$ (properties of similarity, diagram)

Step 3: Get creative!

1 $\angle ADE = \angle CBE$ (Step 2)

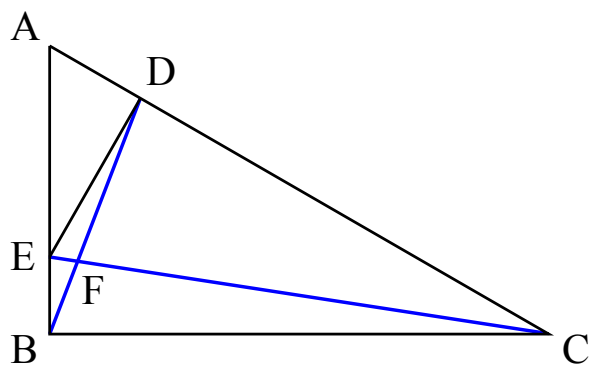
2 AC is a straight line, so $\angle ADE + \angle CDE = 180°$ (diagram, angles along a straight line)

Therefore, $\angle CBE + \angle CDE = 180°$ (substitution)

$\square$

Part 2: Show that $\angle ADB + \angle BEC = 180°$

First, we draw the line segments BD and EC on the digram. Label the intersection of these line segments as point F.



Step 1: Show that $\triangle AEC$ is similar to $\triangle ADB$:

1 $|AD| \times |AC| = |AE| \times |AB|$ or $\dfrac{|AE|}{|AD|} = \dfrac{|AC|}{|AB|}$ (given, algebra)

2 $\angle EAC = \angle DAB$ and the angles are contained (clearly/diagram)

Therefore, $\triangle AEC \sim \triangle ADB$ (SAS)

Step 2: Show that $\angle EBF = \angle DCF$:

1 $\triangle AEC \sim \triangle ADB$ (Step 1)

2 $\angle ACE$ and $\angle ABD$ are corresponding angles of $\triangle AEC$ and $\triangle ADB$ (diagram)

Therefore, $\angle ACE = \angle ABD$ (properties of similarity)

Or, the same result can be written as $\angle EBF = \angle DCF$ (diagram)

Step 3: Show that $\triangle EFB$ is similar to $\triangle DFC$:

1 $\angle EBF = \angle DCF$ (Step 2)

2 $\angle EFB = \angle DFC$ (diagram, opposite angles)

Therefore, $\triangle EFB \sim \triangle DFC$ (AA)

Step 4: Show that $\dfrac{|EF|}{|DF|} = \dfrac{|BF|}{|CF|}$:

1 $\triangle EFB \sim \triangle DFC$ (Step 3)

2 EF with DF, and BF with CF are pairs of corresponding sides of $\triangle EFB$ and $\triangle DFC$ (diagram / match vertices)

Therefore, $\dfrac{|EF|}{|DF|} = \dfrac{|BF|}{|CF|}$ (properties of similarity)

Step 5: Show that $\triangle BCF$ is similar to $\triangle EDF$:

1 $\dfrac{|EF|}{|DF|} = \dfrac{|BF|}{|CF|}$, which means that $\dfrac{|BF|}{|EF|} = \dfrac{|CF|}{|DF|}$ (Step 4, algebra)

2 $\angle DFE = \angle CFB$ (diagram, opposite angles)

Therefore, $\triangle BCF \sim \triangle EDF$ (SAS)

Step 6: Show that $\angle BCE = \angle EDB$:

1 $\triangle BCF \sim \triangle EDF$ (Step 5)

2 $\angle BCF$ and $\angle EDF$ are corresponding angles of $\triangle BCF$ and $\triangle EDF$ (diagram / match vertices)

Therefore, $\angle BCF = \angle EDF$ (properties of similarity)

Or, this result can be written as $\angle BCE = \angle EDB$ (diagram)

Step 7: Get creative!

We have that,

$$\begin{aligned}
\angle BEC &= 180^\circ - (\angle CBE + \angle BCE) && \text{(interior angles of a triangle add to } 180^\circ\text{)} \\
&= 180^\circ - \angle CBE + \angle BCE && \text{(distributive property)} \\
&= \angle CDE + \angle BCE && \text{(Part 1)} \\
&= \angle CDE + \angle EDB && \text{(Step 6)} \\
&= \angle BDC && \text{(digram)} \\
&= 180^\circ - \angle ADB && \text{(diagram, angles along straight line } AC\text{)}
\end{aligned}$$

Therefore, we subtract $\angle ADB$ from both sides to get

$$\angle ADB + \angle BEC = 180^\circ$$

as required.

$\square$