UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

Enriching Mathematics and Computer Science for 50 years

Math Circles: Graph Theory II

Centre for Education in
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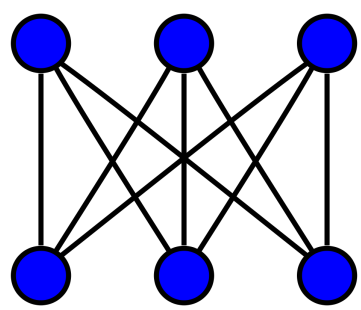
1 Crossing numbers

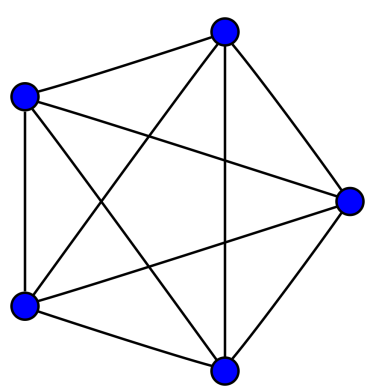
Definition 1.1. The *complete bipartite graph* $K_{m,n}$ is the graph connecting a set of m vertices to a set of n vertices, with every eligible pair of vertices joined by an edge.

Example 1.2. Here is a picture of $K_{3,3}$.



Definition 1.3. The *complete graph* K_n is the graph connecting a set of n vertices to each other, with every eligible pair of vertices joined by an edge.

Example 1.4. Here is a picture of K_5.



Definition 1.5. The *crossing number* of a graph is the minimum number of edge crossings present in any drawing of the graph on the plane.

Example 1.6. From our attempts to solve the water-gas-electricity problem, we conjecture that the crossing number of $K_{3,3}$ is 1, and that the crossing number of K_5 is 1. More generally, we compile the following table of conjectured crossing numbers.

graph	crossing number
K_3	0
K_4	0
K_5	1
K_6	?
$K_{m,2}$	0
$K_{3,3}$	1
$K_{4,3}$	2
$K_{5,3}$	3
$K_{4,4}$	?

Amazingly, the crossing numbers for K_n and $K_{m,n}$ are still not known in general. However, we can show that certain graphs have nonzero crossing number, meaning that they cannot be drawn in a plane without some of the edges crossing.

Theorem 1.7. *K_5 has crossing number* 1.

Proof. It is fairly easy to draw K_5 with one pair of edges crossing. We show that there must be at least one crossing. If there are no crossings, then the graph forms a polyhedron with 5 vertices and 10 edges. From Euler's formula,

$$\begin{aligned} V - E + F &= 2 \\ 5 - 10 + F &= 2 \\ F &= 7 \end{aligned}$$

so the graph has 7 faces. Each face has at least three edges, since the face must be a polygon, and a polygon has at least three edges. Counting the edges around each face, we double-count the edges, since each edge borders two faces. Hence the number of edges in K_5 is at least

$$E \geq \frac{3F}{2} = \frac{21}{2} = 10.5.$$

But this contradicts $E = 10$, so it is not possible to draw K_5 with no edges crossing. □

Theorem 1.8. *$K_{3,3}$ has crossing number* 1.

Proof. Similar to K_5, the graph $K_{3,3}$ has 6 vertices and 9 edges. If it were possible to draw $K_{3,3}$ in the plane without edge crossings, then Euler's formula implies $F = 5$. Unlike K_5, there are no triangles in $K_{3,3}$, since a triangle would imply that there exists an edge within one or the other of the two sets of 3 vertices. Hence each face in $K_{3,3}$ has at least four edges, so

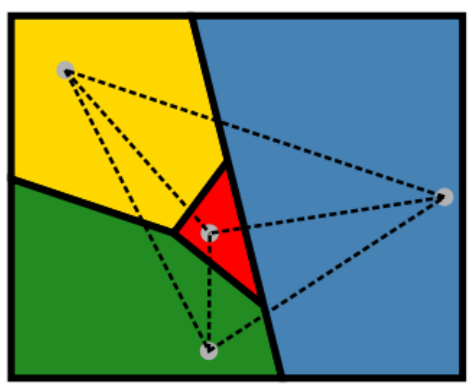
$$E \geq \frac{4F}{2} = \frac{20}{2} = 10,$$

contradicting $E = 9$. □

2 The Four-Colour Theorem

A *map* is a division of the plane into contiguous regions, such as a world map delineating political divisions:

By a *colouring* of a map, we mean an assignment of colours to each region, such that regions sharing a common border segment of nonzero length are assigned different colours. The question we ask is to find the minimum number of different colours required. It is easy to see that there exist maps requiring four colours:



As indicated by the diagram above, we can turn the problem into a graph theory problem, by drawing a vertex in each region, and connecting two regions with an edge if and only if they share a common border segment of nonzero length. There is a celebrated theorem, called the *Four-Colour Theorem*, which states that every graph of crossing number zero can be coloured using at most four colours.

The proof of the Four-Colour Theorem is very hard, but we can prove less optimal bounds with less effort. Here is a proof of the six-colour theorem.

Lemma 2.1. *In any graph with no edge crossings and at least three vertices, we have $E \leq 3V - 6$. (The requirement $V \geq 3$ is necessary: otherwise, the graph could have $V = 2$ and $E = 1$.)*

Proof. As in the proof of Theorem 1.7, we have

$$E \geq \frac{3F}{2}$$

and Euler's formula implies $V - E + F = C + 1 \geq 2$. Putting the two together, we obtain

$$V - E + \frac{2E}{3} \geq 2.$$

Solving for E, we find $E \leq 3V - 6$. □

Theorem 2.2 (Six-Colour Theorem)**.** *Every graph with no edge crossings can be coloured using at most six colours.*

Proof. We prove the theorem by induction. If $V < 6$ then the theorem is obvious. If $V \geq 6$, then there must exist one vertex v of degree at most 5, since otherwise if every vertex had degree at least 6 then

$$E \geq \frac{6V}{2} = 3V,$$

contradicting Lemma 2.1. Delete the vertex v from the graph. By induction, the resulting graph, with v deleted, can be coloured with six colours. Now add v back into the graph. Since v has at most degree 5, and we have six colours, there exists a colour that can be used for v, so the original graph can be coloured using six colours. □