UNIVERSITY OF
WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Senior Math Circles
An Introduction to Inequalities

February 20, 2013

Inequalities in Geometry:

Here are some rudimentary facts:

1) A line is the shortest distance between two points.

2) Thus, we have that any two sides of the nondegenerate $\triangle ABC$ is always bigger than the third. For instance, the straight line from A to B is the shortest possible distance between the two points. Thus, going from A to C then to B takes longer. We have then that $AC + CB > AB$.

This leads to a cool substitution trick for geometric inequalities. If we have that a, b, c are sides of a triangle, then there exist x, y ,z non-negative such that:

$a = x + y$, $b = y + z$, $c = z + x$.

3) We can also get something similar out of the quadrilateral $ABCD$: It's true that $AC + BD > AB + CD$ and also $AC + BD > AD + BC$ (why?)

Now, here's a classical problem that we'll do:

A set of 2n distinct points lie in the plane, n coloured red and n blue. Show that it is possible to connect each red point to a blue point, with no two red points paired with the same blue point, so that no two of the n segments thus created intersect.

Think about generating an algorithm to reach a certain state where no intersection can occur. Remember also to show that we can actually get to this state.

Inequalities in Number Theory:

A trivial but important fact that in number theory is that:

if $a|b$ then $|a| \leq |b|$.

For an easy example, think about why there are very few natural number solutions (a,b) such that $a^3 + b^3 | a^2 + b^2$.

When solving diophantine equations, we also ought to keep in mind something called the squeeze principle.

To illustrate this: find all positive solutions (m,n) such that $m^2 = n^2 + n + 1$.

So, since the RHS looks more informative and promising, we consider it and how it can equate to a square. With a bit of ingenuity, we find that :

$n^2 < n^2 + n + 1 < n^2 + 2n + 1 = (n+1)^2$.

But, that means that the perfect square m^2 is between two consecutive squares n^2 and $(n+1)$. When can this happen? Answer: never.

Inequalities in Algebra:

By far the most commonly used fact one is that $x^2 \geq 0$.

Try to get it into something like this.

Example: $a^3 + b^3 + c^3 - 3abc$.

Think about it in terms of a polynomial and factor it. Substitute in c=k-a-b.

But, often the problem is that we don't know how to get it into this form.

So here are three important techniques to help you with Olympiad inequalities:

1) The most fundamental yet crucial tool you have is the AM-GM inequality; this says that:

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$

Equality case is when all the a_i's are equal.

This can be applied in creative ways.

Example: (CMO 2002) Prove that for all a,b,c $\in \mathbb{R}^+$

$$\frac{a^3}{bc} + \frac{b^3}{ac} + \frac{c^3}{ab} \geq a + b + c$$

2)Rearrangement inequality (related to the Greedy algorithm):

This is a fairly intuitive inequality, essentially we are pairing the biggest with the biggest, the smallest with the smallest to maximise the sum of the products. Not so intuitive is the rearrangment part.

Let $a_1 \leq a_2 \leq ... \leq a_n$, $b_1 \leq b_2 \leq ...b_n$. Let $c_1, c_2, ...c_n$ be a rearrangement of the numbers a_i:

we have that $a_1b_n + a_2b_{n-1} + ... + a_nb_1 \leq c_1b_1 + ... + c_nb_n$.

And of course, the RHS is maximized when $c_i = b_i$.

Now a tool for multi-variable inequalites:

3) Cauchy-Schwarz Inequality:

Let $a_1, a_2, ..., a_n$, $b_1, b_2, .., b_n$ be real numbers, then:

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1b_1 + a_2b_2 + ... + a_nb_n)^2$$

Interestingly, equality holds not only when $a_i = b_i$, but when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$.

Example: find all solutions to:

$$(2013 - a_1)^2 + (a_1 - a_2)^2 + \cdots + (a_{2012} - a_{2013})^2 + a_{2013}^2 = 2013$$

One good way to tell which inequalities to use is to check and find the equality cases.