Theorem. An angle at the circumference of a circle equals one-half the angle at the center subtended by the same arc.

Theorem. Angles on the same arc (or equal arcs) of a circle are equal.
**Theorem.**

Given:

![Diagram of a circle with points P, Q_1, Q_2, R_1, R_2, and O.]  

Then \( PQ_1 \cdot PQ_2 = PR_1 \cdot PR_2 = PO^2 - r^2 \).

**Definition:** This number is called the **power** of \( P \) with respect to the circle.

If the circle is \( \lambda \), write: \( \mathcal{P}(P, \lambda) \)

---

**Theorem.**

Given:

![Diagram of a circle with points P, Q_1, Q_2, T, and O.]  

Then \( PQ_1 \cdot PQ_2 = PT^2 = \mathcal{P}(P, \lambda) \).

**Proof.**

Proof follows from ....
If $P$ is inside the circle:

$$-PQ_1 \cdot PQ_2 = -PR_1 \cdot PR_2 = PO^2 - r^2.$$

This number is again called the power of $P$ with respect to the circle.

The power of $P$ with respect to the circle $\lambda$, $P(P, \lambda)$
- is positive if $P$ is outside $\lambda$
- is negative if $P$ is inside $\lambda$
- if $P$ is on $\lambda$?

**Definition.** The **radical axis** of two circles is the locus of all points $P$ whose power is the same with respect to the two circles.

**Theorem.** The radical axis of two circles is a straight line perpendicular to the line through the centers of the circles.

**Proof.**
Constructing the radical axis.

Big deal about the radical axis:
There is a circle $\mu$ centered at $P$ which cuts circles $\lambda_1, \lambda_2$ at right angles if and only if $P$ is on the radical axis of circles $\lambda_1, \lambda_2$.
Also say $\mu$ is orthogonal to $\lambda_1, \lambda_2$.  

\( P \)

\( T_1 \)

\( T_2 \)

\( O_1 \)

\( O_2 \)

\( \lambda_1 \)

\( \lambda_2 \)

\( \mu \)

radical axis

\( \sim \) radical axis

\( A \)

\( B \)
**Theorem.**

Given

\[ \lambda \]

Then \( PO \cdot P'O = r^2 \)

**Proof**

Points \( P \) and \( P' \) above are said to be **inverses** of each other, with respect to the circle \( \lambda \).

Write \( P' \) for the inverse of \( P \) if the circle is clear. Otherwise write \( I_\lambda(P) \).

\[ P'' = (P')' = P \]

\( O' = ?? \)

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**Theorem.** Let \( \lambda_1, \lambda_2 \) be concentric circles, with radii \( r_1, r_2 \). Then inversion in \( \lambda_1 \) and inversion in \( \lambda_2 \) differ by a similarity: either a contraction or dilation by the factor of \( \frac{r_2}{r_1} \).

\[ \]
**Theorem.** The inverse of a circle \( \mu \) not passing through the center of inversion \( O \) is a circle. The line through the centers of \( \mu \) and \( \mu' \) passes through \( O \).

**Proof**

**Theorem.** The inverse of a line \( l \) not through the center of inversion is a circle passing through the center of inversion.

**Proof**

Diagram is a special case!

Conversely, the inverse of a circle passing through the center of inversion is a line not passing through the center of inversion.
Theorem. The inverse of a line through the center of inversion is that line itself.

Theorem. Inversion is conformal. That is, preserves angles.

Theorem. Suppose $\mu_1$ and $\mu_2$ are non-overlapping circles. Then there is an inversion that transforms them into concentric circles.
Steiner chains of circles.

Steiners Porism. If two non overlapping circles admit one Steiner chain, they admit one of the same length starting with any circle that is between the first two circles, and tangent to both.

Proof.
Invert the two black circles into concentric circles.
The result is then obvious!