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Grade 11/12 Math Circles
Equivalence Relations on Infinite Sets
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Review

A function (or a mapping) f from a set A to a set B is a type of relation (essentially, a subset) on their Cartesian product, $A \times B$. Notationally, this is written

$$f : A \to B$$

A is the **domain set** of the function, B is the **range set**.

f is said to be **injective or one-to-one** if whenever $f(x) = f(y)$, then $x = y$. Informally, every element in the domain is assigned to exactly one element in the range.

f is said to be **surjective or onto** if for all $y \in B$, there exists an $x \in A$ such that $f(x) = y$. Informally, every element in the range has at least one element in the domain which maps to it.

f is said to be **bijective or "in 1-1 correspondence"** if it is both surjective and injective.

Equivalence Relations

An equivalence relation on a set X is a relation $R \subseteq X \times X$, where:

1. $(x, x) \in R$, for all $x \in X$ **(Reflexive Property)**
2. If $(x, y) \in R$, then $(y, x) \in R$ **(Symmetric Property)**
3. If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. **(Transitive Property)**

If $(a, b) \in R$, then we write $a \sim b$, which is read "a is equivalent to b".

Types of Equivalence Relations

1. "=" is an equivalence relation on $\mathbb{N}$, forming the set $R \subseteq \mathbb{N} \times \mathbb{N}$, with $R = \{(a,b) \mid a,b \in \mathbb{N}; (a,b) \in R \Leftrightarrow a = b\}$. To verify this is an equivalence relation, observe that

   (a) $(x,x) \in R$, for all $x \in \mathbb{N}$, since $x = x$ for any natural number x

   (b) If $(x,y) \in R$, then $(y,x) \in R$, since if $x = y$, then $y = x$

   (c) If $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$, since if $x = y$, $y = z$, then $x = z$

2. Let Ω be a set containing some sets. Define a relation $R \subseteq \Omega \times \Omega$ by $R = \{(A,B) \mid A,B \in \Omega; (A,B) \in R \Leftrightarrow$ there exists a bijection $f : A \to B\}$. In other words, two sets are related if there is a bijection between them.

   To see why this defines an equivalence relation, observe that

   (a) $(A,A) \in R$, for all $A \in \Omega$, since the identity function ($f(x) = x$ for all x) is clearly a bijection from A to A.

   (b) If $(A,B) \in R$, then $(B,A) \in R$ - if there is a bijection f from A to B, then there is an inverse function f^{-1} from B to A, which is also a bijection.

   (c) If $(A,B) \in R$ and $(B,C) \in R$, then $(A,C) \in R$. Since $(A,B) \in R$, there is a function g from A to B. Since $(B,C) \in R$, there is a function f from B to C. Then the composition of these functions, $f \circ g$, is a bijection from A to C, with $(f \circ g)(x) = f(g(x))$ where $x \in A$, $g(x) \in B$, $f(g(x)) \in C$. To see this, we need to show $(f \circ g)$ is one-to-one and onto.

      i. One-to-one: If $(f \circ g)(x) = (f \circ g)(y)$, then $f(g(x)) = f(g(y))$. Since f is a bijection, $g(x) = g(y)$. Since g is a bijection, $x = y$. Therefore, if $(f \circ g)(x) = (f \circ g)(y)$, then $x = y$, which shows $(f \circ g)$ is one-to-one.

      ii. Onto: Let $z \in C$. Since there is a bijection from B to C, there is an element $y \in B$ such that $f(y) = z$. Since there is a bijection from A to B, there is an element $x \in A$ such that $g(x) = y \in B$. So then $z = f(y) = f(g(x)) = (f \circ g)(x)$, and $x \in A$. Hence $(f \circ g)(x)$ is onto.

From 2, we can conclude that if there is a bijection from A to B, then A can be viewed as equivalent to B. This gives us the following definition.

Definition (Equivalent Sets)**.** *Let Ω be a set containing some sets. For any two sets (finite or infinite size) $A, B \in \Omega$, we say "A and B are equivalent" if and only if there exists a bijection from A to B. We write $A \sim B$ as before.*

This definition gives us a way to compare sets of both finite and infinite size. It is easy to show or disprove equivalence of finite sets. It is more difficult for infinite sets.

Equivalence of Infinite Sets

Example. $\mathbb{N} \sim$ *Even Integers*

Proof. Observe that both sets are infinite. Your intuition may suggest there are more even integers than natural numbers, but we will show this is not the case under our definition.

We will construct a bijection $f : \mathbb{N} \to$ Even Integers. For any natural number $n \in \mathbb{N}$, define f to be:

$$f(n) = \begin{cases} (n-1) & \text{if } n \text{ is odd} \\ -n & \text{if } n \text{ is even} \end{cases}$$

Informally, {1, 3, 5, 7, 9, ...} are sent to {0, 2, 4, 6, 8, ...} respectively and {2, 4, 6, 8, 10, ...} are sent to {-2, -4, -6, -8, -10 ...} respectively.

To verify this is a bijection, we show

1. Injective: Suppose $f(n_1) = f(n_2)$. By the definition of our function, $(n-1) \neq -n$ for any n. So they must both be either odd or even. If both are odd, $f(n_1) = n_1 - 1 = n_2 - 1 = f(n_2)$ only if $n_1 = n_2$. If both are even, $f(n_1) = -n_1 = -n_2 = f(n_2)$ only if $n_1 = n_2$. Thus, if $f(n_1) = f(n_2)$, then $n_1 = n_2$, so the function is injective.

2. Surjective: Let $y \in$ Even Integers.

   If $y = 0$, then $f(1) = 0 = y$.

   If $y < 0$, then $f(-y) = -(-y) = y$, and note $(-y) \in \mathbb{N}$ and it is even, since y is even.

   If $y > 0$, then $f(y+1)) = ((y+1)-1) = y$, and note $y+1 \in \mathbb{N}$ and $y+1$ is odd since $y > 0$ is even. So for every element in the range, there is an element in the domain which maps to it.

Thus the function is surjective and injective, and hence bijective. Therefore, by our definition, a bijection exists between the two, and so $\mathbb{N} \sim$ Even Integers.

$\square$

Example. *Let $a < b$ and $c < d$ with $a, b, c, d \in \mathbb{R}$. Then the set of all real numbers in the open interval (a, b) is equivalent to the set of all real numbers in the open interval (c, d).*

Proof. We will show that there is a bijective function

$$f : (a, b) \to (c, d)$$

Intuitively, we are basically "scaling" one interval to match the other. For any $x \in (a, b)$, define

$$f(x) = c + \frac{x - a}{b - a}(d - c)$$

It is easily verified that this function is injective. To show it is onto, let $y \in (c, d)$.

Let $x = a + \dfrac{y - c}{d - c}(b - a)$. Since $y > c$, $\dfrac{y - c}{d - c}(b - a) > 0$, so $x = a + \dfrac{y - c}{d - c}(b - a) > a$. Since $y < d$, $\dfrac{y - c}{d - c} < 1$, so $\dfrac{y - c}{d - c}(b - a) < (b - a)$ and hence $x = a + \dfrac{y - c}{d - c}(b - a) < b$.

Therefore, $x \in (a, b)$, and you can check to see $f(x) = y$. $\square$

Example. *The open unit interval $(0, 1)$ is equivalent to $\mathbb{R}$ (!).*

Proof. Define a mapping $f : (0, 1) \to \mathbb{R}$ by

$$f(x) = \tan\left[(2x - 1)\frac{\pi}{2}\right]$$

.

Recall from the properties of trigonometric functions that the tangent function is actually a bijection from the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to the interval $(-\infty, \infty)$.

Observe also that the argument of the tangent in our definition of f, $(2x - 1)\frac{\pi}{2}$, is a bijection between $(0, 1)$ and $(-\frac{\pi}{2}, \frac{\pi}{2})$ (it is just a simplified version of the bijection we used in the previous example).

Then f is just a composition of two bijective functions, and hence it is also a bijection, going from $(0, 1)$ to $(-\infty, \infty)$, i.e. $\mathbb{R}$.

Therefore, $(0, 1) \sim \mathbb{R}$.

The German mathematician Georg Cantor is often credited with laying the foundations of set theory, and in particular, infinite sets.

Proposition. $\mathbb{N}$ *is not equivalent to* $\mathbb{R}$

Since $\mathbb{R}$ is equivalent to $(0,1)$, we will show instead that $\mathbb{N}$ is not equivalent to $(0,1)$.

The proof proceeds as follows. We will first assume that $\mathbb{N} \sim (0,1)$, and then show that this is impossible. If this is impossible, the only other possibility is that $\mathbb{N}$ **is not** equivalent to $(0,1)$, which completes our proof.

For the rest of this proof, we are going to represent any $x \in (0,1)$ using its decimal expansion. That is,

$$x = 0.a_1a_2a_3a_4a_5a_6a_7 \dots a_n \dots$$

where a_i is any digit from 0 to 9 for all i, and at least one a_i is not 9, and at least one a_i is not 0.

Now, suppose $\mathbb{N} \sim (0,1)$. Then there is a bijection from $\mathbb{N}$ to $(0,1)$. So for any real number $x \in (0,1)$, it can be matched to exactly one natural number $n \in \mathbb{N}$. Thus we should be allowed to list out the element in $(0,1)$ in the following fashion.

$$\begin{aligned}
1 &: 0.a_{11}a_{12}a_{13}a_{14}a_{15} \dots a_{1n} \dots \\
2 &: 0.a_{21}a_{22}a_{23}a_{24}a_{25} \dots a_{2n} \dots \\
3 &: 0.a_{31}a_{32}a_{33}a_{34}a_{35} \dots a_{3n} \dots \\
4 &: 0.a_{41}a_{42}a_{43}a_{44}a_{45} \dots a_{4n} \dots \\
&\vdots \\
m &: 0.a_{m1}a_{m2}a_{m3}a_{m4}a_{m5} \dots a_{mn} \dots \\
&\vdots
\end{aligned}$$

Consider the real number $b = 0.b_1b_2b_3b_4b_5 \dots$, where b_i is a digit from 0 to 9 for all $i > 1$, with $b_1 \neq 0$, and $b_i \neq a_{ii}$ for all i. Then clearly, $b \in (0,1)$.

For example, $b_1 \neq a_{11}, b_1 \neq 0$, $b_2 \neq a_{12}$, $b_3 \neq a_{33}$, $b_n \neq a_{nn}$ etc. for all n....

Suppose b were in the listing. This means it would be assigned to some number in the list, say k. So in the list above, b would equal

$$k : 0.a_{k1}a_{k2}a_{k3}a_{k4} \dots$$

Then for any n, $b_n = a_{nn}$. But this is impossible! b was constructed so that $b_n \neq a_{nn}$, for all n. This is a contradiction. Then b is in fact not in the listing given, so the function f cannot be surjective. Therefore, $\mathbb{N}$ is not equivalent to $(0,1)$, and subsequently cannot be equivalent to $\mathbb{R}$. $\square$

Recall

The power set of a set X, denoted $\mathcal{P}(X)$, is the set containing all the subsets of X.

Theorem (Cantor's Theorem)**.** *For any set X, $\mathcal{P}(X)$ is not equivalent to X.*

Proof. If $X = \emptyset$, then $\mathcal{P}(\emptyset) = \{\emptyset\}$. The empty set has no elements; its power set has one element (the empty set). So there cannot be a bijection between the two.

In general, if X is finite, it is obvious; if X has n elements, then its power set has $2^n > n$ elements.

For infinite sets, we will proceed using contradiction. Suppose that if X is infinite, then $X \sim \mathcal{P}(X)$. Then there is a bijection $f : X \to \mathcal{P}(X)$ where any element $x \in X$ is mapped uniquely to a subset of X.

Note that either $x \in f(x)$ or $x \notin f(x)$. Construct the set $D = \{x \in X \mid x \notin f(x)\}$. This is a subset of X and so is in $\mathcal{P}(X)$. Because f is a bijection, there must be $d \in X$ such that $f(d) = D$.

If $d \in D$, then by definition of D, $d \notin D$, a contradiction. If $d \notin D$, then by definition of D, $d \in D$, a contradiction. Both cases are contradictory; therefore, no $x \in X$ can map to $D \in \mathcal{P}(X)$.

But then f is not onto, and hence cannot be a bijection. So in fact, $\mathcal{P}(X)$ is not equivalent to X. $\square$