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Grade 11/12 Math Circles
Mathematical Induction
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INTRODUCTION

Mathematical Induction (or simply, "induction") is a method of proving statements about the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$. Suppose that $P(n)$ is a statement about the number n. Suppose we want to prove that $P(n)$ is true for all $n \in \mathbb{N}$. The way induction works is as follows:

- Prove that $P(1)$ is true, AND
- Prove the statement "If $P(k)$ is true for some $k \in \mathbb{N}$, then $P(k+1)$ must also be true."

If we do both of the above, then we can conclude that $P(n)$ is true for all $n \in \mathbb{N}$. The first step is usually called the "base case", and the second step is usually called the "induction step". The premise $P(k)$ is oftern called the "induction hypothesis". We will see how do use induction in practice, but first, we should make sure we understand the concept, and WHY it works, because it is not so obvious if it is a new concept to you.

There are a couple of analogies which might help you to understand the idea behind induction.

If I have dominoes standing up in a pattern, and I want to make sure to knock them all down, I really need to ensure only two things:

- that I knock the first domino over, AND
- that each domino is close enough to the next one, so that if any domino falls, it will knock over the one after it.

These two things together will certainly ensure that all of them will eventually get knocked down. As you can probably tell, this is the same concept as the above description of induction, and it is even typed in the same way, so that the similarity is more apparent.

Suppose I want to get a secret message to many people all sitting next to each other. If I

- tell the first person the secret, AND
- get everyone to promise that if they hear the secret, they will tell the person after them,

then I can be sure that the secret will eventually reach everyone.

In both of these examples, I have initiated something, and made sure that this little system or machine will perpetuate itself after I set it in motion. And that's what induction does. The only difference with mathematical induction is that we don't need to wait for it to happen "eventually", because the logic of it means it all just holds for all of the infinitely many natural numbers.

So, to break it down, if I can prove the induction step, that $P(k)$ will lead to $P(k+1)$, then the logic works as follows:

1. If I prove that $P(1)$ is true (the base case), then (if the induction step is proven) $P(1+1) = P(2)$ is also true (using $k = 1$ in the above induction step).
2. Now that I know that $P(2)$ is true, I can re-apply the induction step result (with $k = 2$) to conclude that $P(2+1) = P(3)$ is true as well.
3. Now that $P(3)$ is true, it leads to $P(3+1) = P(4)$ being true.

You can see how this pattern continues. It will (eventually, but we don't have to wait!) lead to $P(n)$ being true for ALL $n \in \mathbb{N}$.

EXAMPLES

If you understand the above concept, then that is awesome. Understanding the concept is certainly required to be able to actually use induction to prove things, but actually applying it prove things takes some work, and these examples show the line of thinking to use induction. Essentially, what usually happens is that the base case is pretty easy to show, and then you need to prove the induction step. To do so, you assume that $P(k)$ is true (for some arbitrary $k \in \mathbb{N}$, and then prove that $P(k+1)$ will also be true. Usually proving $P(k+1)$ involves the result that $P(k)$ is true. We will see this in the examples:

Example 1: Prove that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Proof: The statement $P(n)$ that we need to prove for all $n \in \mathbb{N}$ is "$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$".

Base case: $P(1)$ is the statement $1 = \frac{1(1+1)}{2}$. It is easy to see that this is true, as the right-hand side quickly reduces to 1, which is equal to the left-hand side. Thus $P(1)$ holds.

Induction hypothesis: Assume that $P(k)$ holds true for some $k \in \mathbb{N}$. Then this means that $1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$. Now we need to somehow prove that $P(k+1)$ is also true. That is, we need to prove that $1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2}$.

Well, observe that the left-hand side can be written as

$$1 + 2 + 3 + \cdots + (k+1) = (1 + 2 + 3 + \cdots + k) + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

(this last equality is because we are assuming the induction hypothesis to be true). Then we work with this a little, putting things over a common denominator:

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

Now looking at the extreme left-hand side and the extreme right-hand side, we see that we have proven $P(k+1)$, that $1+2+3+\cdots+(k+1) = \frac{(k+1)((k+1)+1)}{2}$.

Now that we have proven the induction step, and the base case, we conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$. That is, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

This above example was a little more wordy than needed, because it is meant as an introduction, to you, to how to prove things by induction. Let's do this a little more concisely:

Example 2: Suppose that x and y are integers. Prove that $x^{2n-1}+y^{2n-1}$ has a factor of $(x+y)$ for all $n \in \mathbb{N}$.

Proof:

Base case: For $n=1$, $x^{2n-1}+y^{2n-1} = x^{2-1}+y^{2-1} = x+y$, which clearly has a factor of $(x+y)$ (since it IS $x+y$ itself!).

Induction hypothesis: Suppose that $x^{2k-1}+y^{2k-1}$ has a factor of $(x+y)$, for some $k \in \mathbb{N}$.

Then

$$\begin{aligned} x^{2(k+1)-1}+y^{2(k+1)-1} = x^{2k+1}+y^{2k+1} &= x^2(x^{2k-1}+y^{2k-1}) - x^2y^{2k-1}+y^{2k+1} \\ &= x^2(x^{2k-1}+y^{2k-1}) - y^{2k-1}(x^2-y^2), \end{aligned}$$

which, we can now see, has a factor of $(x+y)$, since the first term is a multiple of $(x^{2k-1}+y^{2k-1})$, which we assumed to have a factor of $(x+y)$, by the induction hypothesis, and the second term is a difference of squares, which includes an $(x+y)$ term.

Thus, by induction, $x^{2n-1}+y^{2n-1}$ has a factor of $(x+y)$ for all $n \in \mathbb{N}$.

CAREFUL!

You can get some weird results if your logic isn't completely sound. For example, try to figure out what is wrong with the below "proof".

Example 3: Prove that all horses are the same colour.

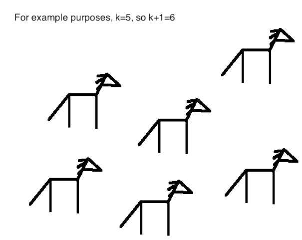
Proof: We can re-state this as:

Prove that $P(n)$ is true for all $n \in \mathbb{N}$, where $P(n)$ means "in any set of n horses, all n horses have the same colour".

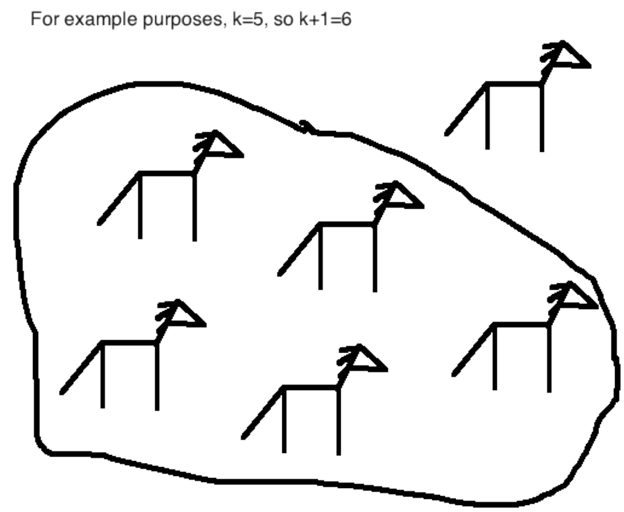
Base case: Consider a set of 1 horse. Clearly, "all" horses in the set have the same colour, since there is only 1 horse. Thus, $P(1)$ holds.

Induction hypothesis: Suppose that for any set of k horses, all k horses have the same colour.

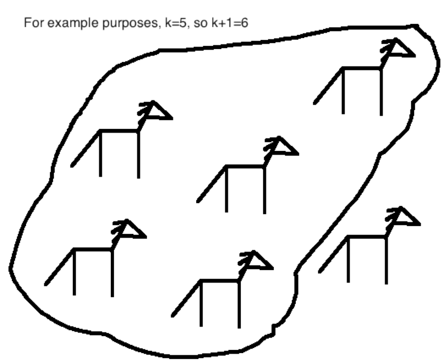
Now consider a set of $k+1$ horses, as in the picture (we have taken $k=5$ for example purposes, so that we deal with a set of $k+1=6$ horses).



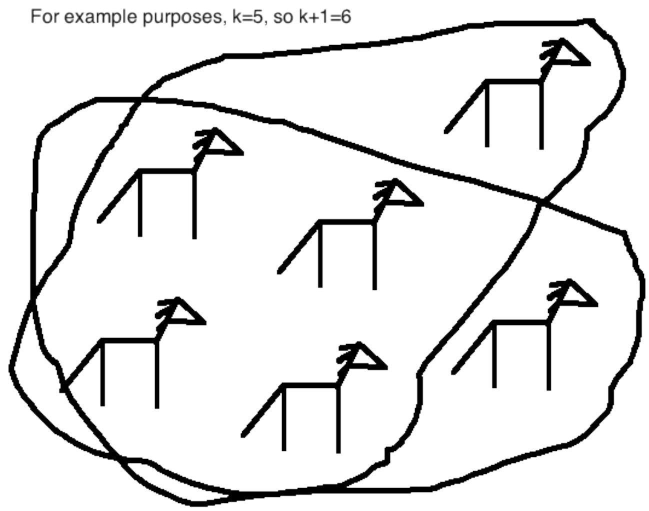
If we remove one horse, we get a set of k horses, which, by the induction hypothesis, must all have the same colour. We don't yet know about that one we removed, though (the top-right horse in the picture).



Well, put that horse that you took out back, and remove a DIFFERENT horse, so that you have (again) a set of k horses. Now THESE k horses must also have the same colour, again by the induction hypothesis.



But now, the horse you put back has to have the same colour as all the horses that remained when you removed it, *but so does the newly-removed one.* That is (as far as the picture goes), the top right horse is the same colour as the four horses that were included both times, and the bottom right horse has the same colour as the four horses that were included both times. Clearly then the top right horse, and the bottom right horse must have the same colour (which is the same colour as the other four horses), so the set of $k+1$ horses all have the same colour.



By induction, any set of n horses must all have the same colour, for all $n \in \mathbb{N}$.

OK! Obviously, all horses don't have the same colour. So what was the problem?

The induction step assumed that there were enough horses to actually remove a different horse, and still have overlap between the remaining horses.

We only proved the base case, of a set of 1 horse. If we tried to take that first step that induction does for us (to conclude that a set of 2 horses must have the same colour) we take away a horse, and the leftover are all one colour, and we take away a different horse, and the leftover is all one colour, but there is no overlap to connect the two removed horses. The picture (while very nice) was misleading! To be able to use that logic, we'd need to have proven the base case for $n = 2$, which would be impossible, since we already know that if we take a set of 2 horses, there is no guarantee that they'll be the same colour!