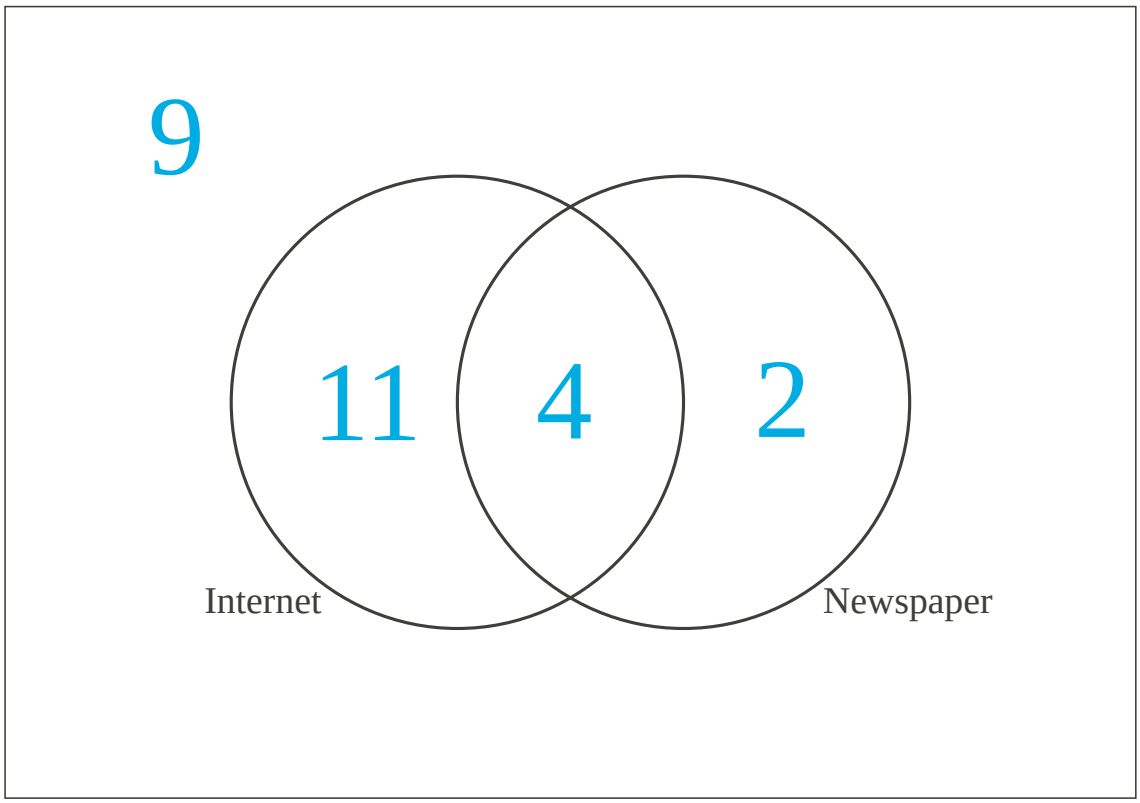
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles
March 2, 2011
Counting

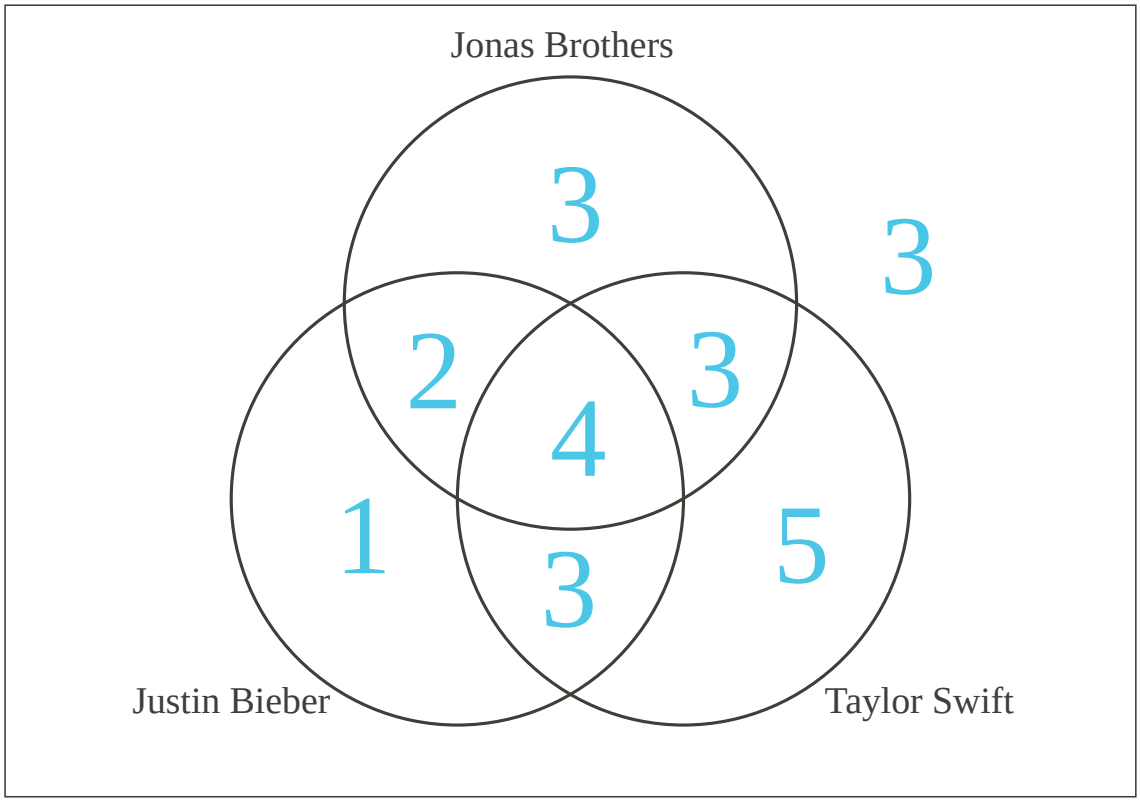
Venn Diagrams

Example 1: Ms. Daly surveyed her class of 26 students to see how they read about the news. 6 of them read the newspaper, and 15 of them read news from the internet. If 4 of the students read both the newspaper and from the internet, how many students in Ms. Daly's class read news from the internet **or** newspaper? How many students don't read from either of these two sources?



From the diagram, we can see that $11 + 4 + 2 = 17$ students read news from the internet or newspaper and 9 students don't read news from either of the two sources.

Example 2: Ms. Iockis surveyed her class of 24 students about the music they like. Fill in the Venn diagram and answer the questions.



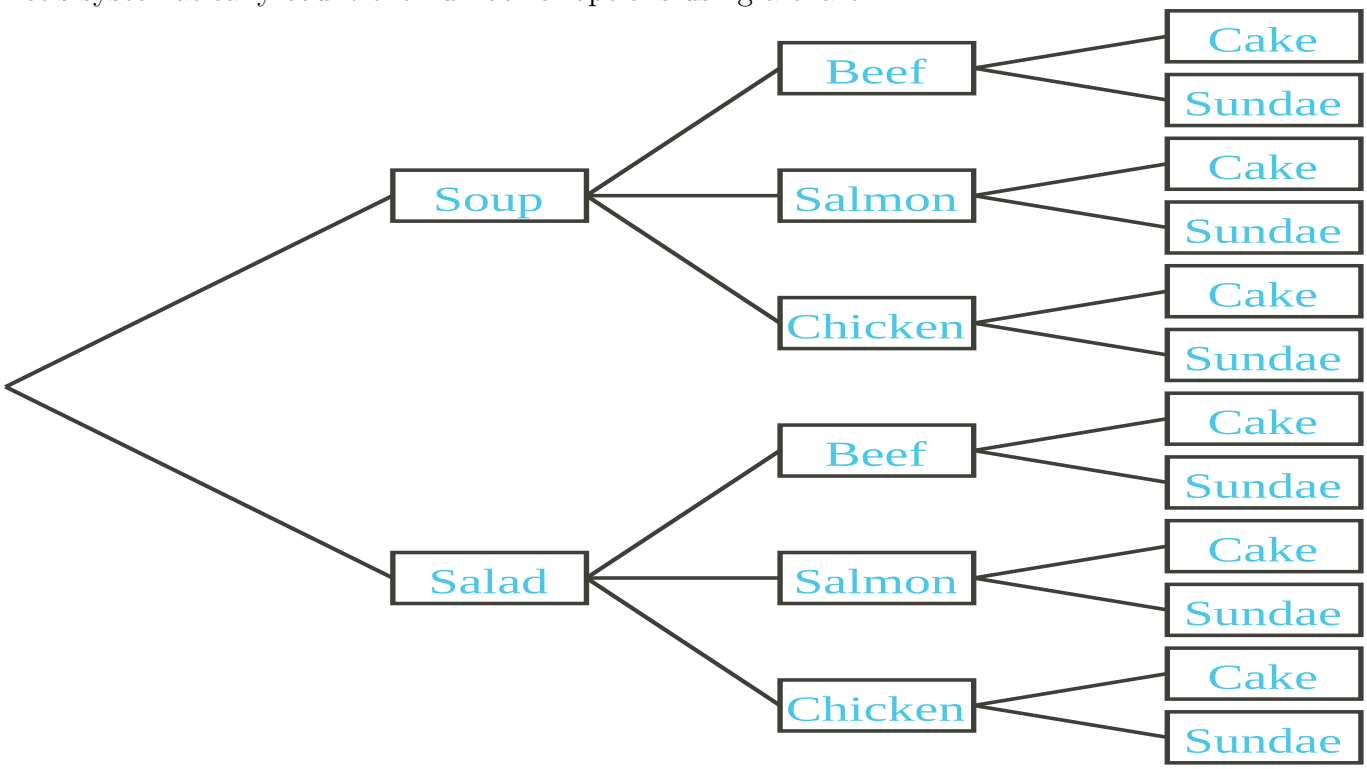
- 15 students like Taylor Swift
- 10 students like Justin Bieber
- 12 students like Jonas Brothers
- 3 students like Justin Bieber and Taylor Swift, but not Jonas Brothers
- Half of the students who like Jonas Brothers also like Justin Bieber
- 20 students like Jonas Brothers or Taylor Swift (or both)
- 8 students like Taylor Swift but not Justin Bieber

How many students don't like any of the three artists? 3
How many students like Jonas Brothers or Justin Bieber? 16
How many students like Jonas Brothers but not Taylor Swift? 5
How many students like Taylor swift but not Jonas Brothers? 8

Fundamental Counting Principle

Problem: You and your family are ordering the Winter special at a restaurant. The menu let's you choose one of two mouth-watering appetizers (soup of the day or salad), one of three hot and juicy entrees (beef steak, salmon or chicken) and one of two scrumptious desserts (cake or sundae). How many possible ways can you have your meal?

Let's systematically count the number of options using a chart.



$\therefore$ There are 12 possible ways to have your meal.

What if there are 10 different appetizers, 15 different entress and 4 different desserts?

This leads us to the **Fundamental Counting Principle** (or the "multiplication rule"), which states: If we have m ways of doing something and n ways of doing some other thing, then there are mn ways of doing both actions.

$10 \times 15 \times 4 = 600$ $\therefore$ There would be 600 different ways!

Exercises:

1. Danny has 3 pairs of glasses, 4 hats and 2 nose rings. He can only wear one of each kind of accessory at a time. How many different "looks" can he have before he has to repeat a "look"?

   $3 \times 4 \times 2 = 24 \therefore$ There are 24 different looks

2. Jillian has a purple six-sided die (each face numbered 1 through 6) and a yellow coin (heads on one side and tails on the other). She flips the coin and rolls the die. How many different possibilities should she expect?

   $6 \times 2 = 12 \therefore$ She should expect 12 outcomes.

3. A robber discovers a safe with a number pad (0 to 9). The safe opens when the correct 4-digit password is entered. How many possibilities are there?

   $10 \times 10 \times 10 \times 10 = 10000 \therefore$ There are 10000 possibilities.

4. You take a coin, flip it and record which side faces up. You continue doing so, recording which side faces up as well as the trial number (e.g. trial 1, trial 2, etc...). How many possibilities in this experiment are there if there are 3 trials? 10 trials? n trials?

   If there are 3 trials, there would be $2 \times 2 \times 2 = 2^3 = 8$ possibilities.
   If there are 10 trials, there would be $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10} = 1024$ possibilities.
   Following this pattern, if there are n trials, there would be 2^{n} possibilities.

5. You number five balls 1 to 5. and put them in a box. You randomly pick out a ball, record the number then put it back in and repeat the process three times. How many possible sequences do you have?

   $5 \times 5 \times 5 = 125 \therefore$ You have 125 possible sequences.

Permutations: Order Matters!

A **permutation** is an arrangement of a certain number of things. For example, if you want to arrange three letters $A, B,$ and C, a CAB is one permutation and ABC, is another permutation. A common question to ask is "given a certain set of things, how many permutations are there?"

Problem: Allie, Bo, Cindy and David are lining up to buy tickets for a basketball game. How many different ways can they line up?

Let's try to list the possibilities out.

ABCD	ABDC	ACBD	ACDB	ADBC	ADCB
BCDA	BDCA	CBDA	CDBA	DBCA	DCBA
CDAB	DCAB	BDAC	DBAC	BCAD	CBAD
DABC	CABD	DACB	BACD	CADB	BADC

$\therefore$ They can line up in 24 different ways.

Q: Is there some systematic way to explain this?
A: Yes there is!!

In the first spot, there are 4 possibilities: Alice, Bo, Cindy or David. Let's say Bo took the first spot.

In the second spot, there are three possible choices left: Alice, Cindy, and David. Let's say David took the second spot.

This leaves us with two choices for the third spot: Alice and Cindy. Let's say Cindy took the third spot.

This leaves us with only one choice left for the last spot.

Using the fundamental counting principle, we get $4 \times 3 \times 2 \times 1 = 24$ different ways to line up.

Factorial Notation: Consider the line up example, but instead of having four people, we have 100 people!! Calculating that even with our method is going to be difficult. Thankfully, we have a notation for this kind of multiplication called the **factorial**.

$$n! = n \times (n-1) \times (n-2) \times ... \times 3 \times 2 \times 1$$

The "!" is the factorial notation, and we say "n!" as "n factorial".

Example: Evaluate 5!
$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Exercises:

6. You are stacking 6 distinct coins on top of each other. How many different arrangements can you have?

   You can have $6! = 720$ different arrangements.

7. You need to do laundry, cook, do homework and hang out with friends. Assuming you cannot multitask, how many ways can you order your tasks?

   You can have $4! = 24$ ways of doing your tasks.

Problem Set

1. (a) How many permutations are there of the letters of the word *guitar*?

   (b) How many of those permutations start with g?

   (c) How many of the permutations in part (a) have t and r together?

2. How many 4-digit numbers are there with no repeating digits? *(Remember: numbers starting with 0 don't count!)*

3. How many 4-digit numbers are there with at least a digit repeated? *(Hint: how many 4 digit numbers are there in total?)*

4. How many 4-digit odd numbers are there with no repeating digits?

5. Mark, Jane, Austin, Mike, Natalie, Laura, Steve and Nick are lining up for photos, but Mike and Jane don't want to be beside each other, and Laura wants to be beside Mark. How many ways can they line up?

6. **[Challenge]** How many different ways can you arrange letters of the word $MISSISSIPPI$?

7. **[Challenge]** You just bought 5 really cool paintings, but you only have space to put 3 of them up on your wall! How many ways can you put the paintings up on your wall assuming you don't care about the order that they're in?

Answers:

1. (a) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

   (b) Since g needs to be in the first spot, there is only one choice to pick the first letter. $1 \times 5 \times 4 \times 3 \times 2 \times 1 = 120$

   (c) If we think of gr as one letter, then we would be trying to arrange 5 letters; 5!=120. However, we could also think of rg as one letter, and there would be another $5! = 120$ arrangements. $2 \times 120 = 240$

2. Since the first spot of the number cannot be 0, we can choose pick 1 to 9. After we used that number for the digit, we have 8 more numbers left **and** 0, so we can pick from 9 different numbers for the second digit. For the third digit, we then have 8 numbers to pick from, and 7 numbers to pick from for the fourth digit. $9 \times 9 \times 8 \times 7 = 4536$

3. First of all, there are $9 \times 10 \times 10 \times 10 = 9000$ 4-digit numbers. Out of all 4-digit numbers, any number either has no repeated digits or has at least one repeated digit. If we subtract the number of 4-digit numbers with no repeated digits from the total number of 4-digit numbers, then what we are left with is the number of 4-digit numbers with at least one repeated digit. $9000 - 4536 = 4464$

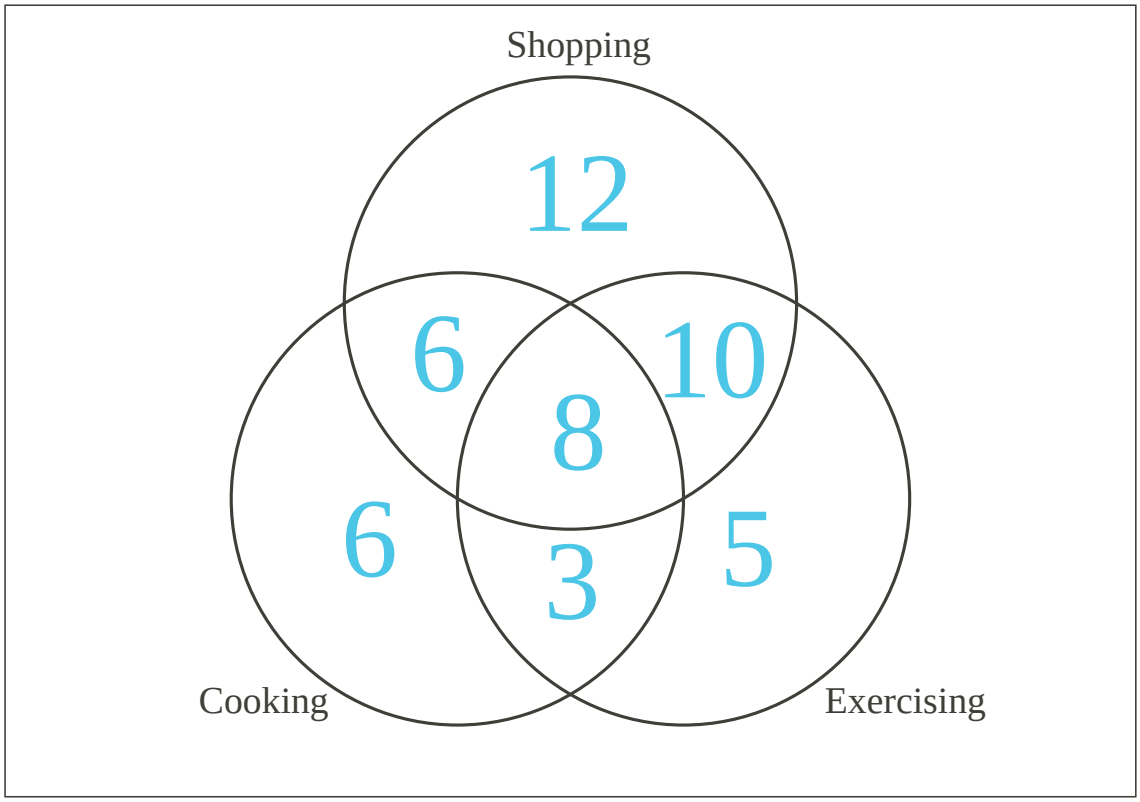
4. We are looking for odd numbers, so we know that the ones digit has 5 choices to pick from (1, 3, 5, 7, or 9). Because we we cannot have repeated digits, the first spot cannot be 0 or the odd number in the unit digit. Thus, we have 8 choices to pick from for the first spot, 8 for the second (because we can now pick 0), 7 for the third and 5 for the last digit. $8 \times 8 \times 7 \times 5 = 2240$

5. Like question 1c, lets pair up Laura and Mark. We then 7 spots to deal with, so there are $2 \times 7! = 10080$ different lineups. Like, question 3, we can count the number of lineups that pair Mike and Jane together, and take those lineups away from 10080. If Mike and Jane are paired up, we would have 6 spots. We have Mark and Laura beside each other, so we multiply by 2, but we also paired Jane and Mike together, so we have to multiply by another 2. $2 \times 2 \times 6! = 2880$. $10080 - 2880 = 7200$

6. 34650 (material covered next week)

7. 10 (material covered next week)

Challenge Problem: A group of students was surveyed about their hobbies. With the help of the clues, complete the Venn diagram.



- Nobody dislikes all three hobbies
- 36 students like to go shopping
- 26 students like to exercise
- There are 44 students that like to shop or exercise
- There are twice as many students who only like to shop compared to the number of students who only like to cook
- Out of the students who don't like to exercise, half of that group likes to cook
- 16% of the whole group of students like all three hobbies
- Out of the students who like more than one hobby, two thirds of that group like shopping and exercising