Series

Brian Forrest

October 20, 2010
A Problem:

Problem:
Problem: Can one perform infinitely many tasks in a finite amount of time?
The Paradox of Achilles:

Paradox of Achilles

- Achilles races a tortoise

Achilles can never catch the tortoise!!!
The Paradox of Achilles:

Achilles races a tortoise who is given a head start.

Achilles reaches where the tortoise began and the tortoise has moved ahead.

Achilles reaches the new point. Again the tortoise has moved ahead.

Achilles reaches this next point, and the tortoise has moved ahead.

And so on ad infinitum.

Conclusion: Achilles can never catch the tortoise!!!
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And so on

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- And so on Ad infinitum.
- **Conclusion:** Achilles can never catch the tortoise!!!
Resolving The Paradox of Achilles:

Resolving the Paradox:

A

T

\[ d_1 = \text{distance traveled in stage 1} \]
\[ t_1 = \text{time to complete stage 1} \]
\[ d_2 = \text{distance traveled in stage 2} \]
\[ t_2 = \text{time to complete stage 2} \]
\[ d_n = \text{distance traveled in stage } n \]
\[ t_n = \text{time to complete stage } n \]

Time to catch the Tortoise = \[ t_1 + t_2 + \cdots + t_n + \cdots \]
\[ = \infty \]
Resolving the Paradox:
We will call each time Achilles moves from a point
Resolving the Paradox:

We will call each time Achilles moves from a point to where the tortoise was
Resolving the Paradox: We will call each time Achilles moves from a point to where the tortoise was a stage.
Resolving The Paradox of Achilles:

Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was a *stage*.

\[ d_1 = \]
Resolving The Paradox of Achilles:

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We will call each time Achilles moves from a point to where the tortoise was a stage.

▶ \( d_1 = \) distance traveled in stage 1
Resolving The Paradox of Achilles:

Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was a stage.

\[ d_1 = \text{distance traveled in stage 1} \Rightarrow t_1 = \]

\[ d_2 = \text{distance traveled in stage 2} \Rightarrow t_2 = \]

\[ d_n = \text{distance traveled in stage } n \Rightarrow t_n = \]

Time to catch the Tortoise
\[ t_1 + t_2 + \cdots + t_n + \cdots = \infty \]
Resolving The Paradox of Achilles:

Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was a *stage*.

- $d_1 = \text{distance traveled in stage 1} \Rightarrow t_1 = \text{time to complete stage 1}$
Resolving The Paradox of Achilles:

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We will call each time Achilles moves from a point to where the tortoise was a stage.

- $d_1 = \text{distance traveled in stage 1} \Rightarrow t_1 = \text{time to complete stage 1}$
- $d_2 =$
Resolving The Paradox of Achilles:

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We will call each time Achilles moves from a point to where the tortoise was a stage.

- \( d_1 = \) distance traveled in stage 1 \( \Rightarrow t_1 = \) time to complete stage 1
- \( d_2 = \) distance traveled in stage 2
Resolving The Paradox of Achilles:

Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was a stage.

- $d_1 =$ distance traveled in stage 1 $\Rightarrow t_1 =$ time to complete stage 1
- $d_2 =$ distance traveled in stage 2 $\Rightarrow t_2 =$
Resolving the Paradox: We will call each time Achilles moves from a point to where the tortoise was a stage.

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We will call each time Achilles moves from a point to where the tortoise was a stage.

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- \( d_2 = \text{distance traveled in stage 2} \Rightarrow t_2 = \text{time to complete stage 2} \)
- \( d_n = \)
Resolving The Paradox of Achilles:

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We will call each time Achilles moves from a point to where the tortoise was a stage.

- $d_1 =$ distance traveled in stage 1 $\Rightarrow t_1 =$ time to complete stage 1
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- $d_n =$ distance traveled in stage n
Resolving the Paradox: We will call each time Achilles moves from a point to where the tortoise was a stage.

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Time to catch the Tortoise
Resolving The Paradox of Achilles:

Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was a stage.

- \( d_1 \) = distance traveled in stage 1 ⇒ \( t_1 \) = time to complete stage 1
- \( d_2 \) = distance traveled in stage 2 ⇒ \( t_2 \) = time to complete stage 2
- \( d_n \) = distance traveled in stage \( n \) ⇒ \( t_n \) = time to complete stage \( n \)

Time to catch the Tortoise = \( t_1 \)
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We will call each time Achilles moves from a point to where the tortoise was a stage.

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- d_n = \text{distance traveled in stage n} \Rightarrow t_n = \text{time to complete stage n}

Time to catch the Tortoise = $t_1 + t_2$
Resolving The Paradox of Achilles:

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We will call each time Achilles moves from a point to where the tortoise was a stage.

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- $d_n = \text{distance traveled in stage } n \Rightarrow t_n = \text{time to complete stage } n$

\[
\text{Time to catch the Tortoise} = t_1 + t_2 + \cdots + t_n
\]
Resolving the Paradox: We will call each time Achilles moves from a point to where the tortoise was a stage.

- $d_1 = \text{distance traveled in stage 1} \Rightarrow t_1 = \text{time to complete stage 1}$
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Time to catch the Tortoise $= t_1 + t_2 + \cdots + t_n + \cdots$
Resolving The Paradox of Achilles:

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- $d_n = \text{distance traveled in stage n} \Rightarrow t_n = \text{time to complete stage n}$

Time to catch the Tortoise

$$= t_1 + t_2 + \cdots + t_n + \cdots$$

$$= \infty ?$$
Problem:
Problem: Can we add infinitely many numbers at the same time?
**Problem:** Can we add infinitely many numbers at the same time?
More precisely,
**Problem:** Can we add infinitely many numbers at the same time? More precisely, given a sequence \( \{a_n\} \),
Problem: Can we add infinitely many numbers at the same time? More precisely, given a sequence \( \{a_n\} \), we can form the formal sum

\[ a_1 + a_2 + a_3 + \cdots \]
**Problem:** Can we add infinitely many numbers at the same time? More precisely, given a sequence \( \{a_n\} \), we can form the *formal sum*

\[
a_1 + a_2 + a_3 + \cdots := \sum_{n=1}^{\infty} a_n.
\]
**Problem:** Can we add infinitely many numbers at the same time? More precisely, given a sequence \( \{a_n\} \), we can form the *formal sum*

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which we call a *series.*
**Problem:** Can we add infinitely many numbers at the same time? More precisely, given a sequence \( \{a_n\} \), we can form the *formal sum*

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**Question:**
**Problem:** Can we add infinitely many numbers at the same time? More precisely, given a sequence \( \{a_n\} \), we can form the *formal sum*

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a_1 + a_2 + a_3 + \cdots := \sum_{n=1}^{\infty} a_n.
\]

which we call a *series*.

**Question:**
What does this formal sum represent?
Problem: Can we add infinitely many numbers at the same time? More precisely, given a sequence \( \{a_n\} \), we can form the \textit{formal sum}

\[
a_1 + a_2 + a_3 + \cdots := \sum_{n=1}^{\infty} a_n.
\]

which we call a \textit{series}.

Question:
What does this formal sum represent? Does it have a value?
Example:
Example: What is
Example: What is

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots \]
Example: What is

\[
\frac{1}{2} + \frac{1}{4} + \cdots
\]
**Example:** What is

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots
\]
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\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \]
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\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \]
Example: What is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$$
**Example:** What is

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots
\]
**Example:** What is

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\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots
\]

\[
= \frac{1}{2} + \cdots
\]
Introduction to Series

**Example:** What is

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots
\]

\[= \frac{1}{2} + \frac{1}{2^2} + \cdots\]
Example: What is

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots
\]

\[
= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots
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\[
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\]

\[
= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} + \cdots
\]
Geometric Interpretation:
Introduction to Series

Geometric Interpretation:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \cdots
\]
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Geometric Interpretation:

\[ \frac{1}{2} + \frac{1}{4} + \cdots = 1? \]
Introduction to Series

Geometric Interpretation:

\[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1?\]
Introduction to Series

Geometric Interpretation:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots
\]
Introduction to Series

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Introduction to Series

Geometric Interpretation:

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots = 1? \]
Introduction to Series

Geometric Interpretation:

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots = 1$
Introduction to Series

Geometric Interpretation:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + ...
\]

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Geometric Interpretation:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \ldots
\]
Geometric Interpretation:

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots \]
Introduction to Series

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\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} + \cdots
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Introduction to Series

Geometric Interpretation:

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\]

= 1?
Introduction to Series
Introduction to Series

\[ \frac{1}{2} \]

**Sum**  
\[ \frac{1}{2} \]

**Area**  
\[ 1 - \frac{1}{2} \]
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\[ \frac{1}{2} + \frac{1}{4} \]

Sum

\[ 1 - \frac{1}{4} \]

Area
Introduction to Series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8}
\]

Sum

\[1 - \frac{1}{8}\]

Area
Introduction to Series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}
\]

Sum

\[
1 - \frac{1}{16}
\]

Area
Introduction to Series

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \]

Sum

\[ 1 - \frac{1}{32} \]

Area
Series:

Introduction to Series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64}
\]

Sum

\[
1 - \frac{1}{64}
\]

Area
Introduction to Series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} = 1 - \frac{1}{128}
\]

Sum

Area

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Introduction to Series

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}
\]

\[
\text{Sum} \quad 1 - \frac{1}{256}
\]

\[
\text{Area}
\]
Introduction to Series

Series:

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256} + \frac{1}{512}
\]

Sum

\[
1 - \frac{1}{512}
\]

Area
Introduction to Series

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} \]

\[ + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} \]

\[ = 1 - \frac{1}{1024} \]

Sum

Area
Introduction to Series

\[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots + \frac{1}{2^k} \]

\[ \text{Sum} \]

\[ 1 - \frac{1}{2^k} \]

\[ \text{Area} \]
Introduction to Series

Note:
Note:

\[ \lim_{k \to \infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} \]
Introduction to Series

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\[ \lim_{k \to \infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = \lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{2^k} \]
Introduction to Series

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\]
Introduction to Series

Note:

\[
\lim_{k \to \infty} \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k} = \lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{2^k} = \lim_{k \to \infty} 1 - \frac{1}{2^k} = 1
\]
Convergence for Series

Definition: [Convergent Series]
Given a sequence \( \{a_n\} = \{a_1, a_2, a_3, \ldots\} \),
Convergence for Series

Definition: [Convergent Series]
Given a sequence \( \{a_n\} = \{a_1, a_2, a_3, \ldots\} \), we define the \( kth \) partial sum, \( S_k \), as

\[
S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n
\]

We say that the series \( \sum_{n=1}^{\infty} a_n \) converges if the sequence of partial sums \( \{S_k\} \) converges. In this case, we write \( \sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k \).

Otherwise, we say that the series diverges and the sum has no defined value.
Definition: [Convergent Series]
Given a sequence \( \{ a_n \} = \{ a_1, a_2, a_3, \ldots \} \), we define the \( k \)th partial sum, \( S_k \), as

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\]
Convergence for Series

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Given a sequence \( \{a_n\} = \{a_1, a_2, a_3, \ldots\} \), we define the \textit{kth partial sum}, \( S_k \), as

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Given a sequence \( \{a_n\} = \{a_1, a_2, a_3, \ldots\} \), we define the \( k \text{th partial sum} \), \( S_k \), as

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\]

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Convergence for Series

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Given a sequence \( \{a_n\} = \{a_1, a_2, a_3, \ldots\} \), we define the \( k \text{th partial sum}, S_k \), as

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\]

We say that the series \( \sum_{n=1}^{\infty} a_n \) converges if the sequence of partial sums \( \{S_k\} \) converges.

In this case, we write

\[
\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k
\]
Convergence for Series

**Definition: [Convergent Series]**

Given a sequence \( \{a_n\} = \{a_1, a_2, a_3, \ldots\} \), we define the *kth partial sum*, \( S_k \), as

\[
S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n.
\]

We say that the series \( \sum_{n=1}^{\infty} a_n \) *converges* if the sequence of partial sums \( \{S_k\} \) converges.

In this case, we write

\[
\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k
\]

Otherwise, we say that the series *diverges*.
Convergence for Series

**Definition: [Convergent Series]**
Given a sequence \( \{a_n\} = \{a_1, a_2, a_3, \ldots\} \), we define the *kth partial sum*, \( S_k \), as

\[
S_k = a_1 + a_2 + \cdots + a_k = \sum_{n=1}^{k} a_n.
\]

We say that the series \( \sum_{n=1}^{\infty} a_n \) *converges* if the sequence of partial sums \( \{S_k\} \) converges.

In this case, we write

\[
\sum_{n=1}^{\infty} a_n = \lim_{k \to \infty} S_k
\]

Otherwise, we say that the series *diverges* and the sum has no defined value.
Suppose \( n = 1 \). We know that
\[
S_k = \sum_{n=1}^{k} \frac{1}{n} = 1 - \frac{1}{k} 
\]converges with
\[
\sum_{n=1}^{k} \frac{1}{n} = 1.
\]
Example:

Suppose $a_n = \frac{1}{2^n}$. 

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Suppose \( a_n = \frac{1}{2^n} \).

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Hence, \( \sum_{n=1}^{k} \frac{1}{2^n} \) converges.
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Hence, $\sum_{n=1}^{k} \frac{1}{2^n}$ converges with

$$\sum_{n=1}^{k} \frac{1}{2^n} = 1.$$
Why use limits?

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\[
a_1 + a_2 + a_3 + \cdots = 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots
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Our result is ambiguous; the "sum" changes if we change the way we parenthesize the terms.
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Why use limits?

Observe:

\[ S_1 = 1 \]

\[ S_2 = 1 - 1 = 0 \]

\[ S_3 = 1 - 1 + 1 = 1 \]

\[ S_4 = 1 - 1 + 1 - 1 = 0 \]

We get

\[ S_k = 1 + (-1) + 1 + \cdots + (-1)^{k+1} = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} \]

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\begin{align*}
S_1 &= 1 \\
S_2 &= 1 - 1 = 0 \\
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S_4 &= 1 - 1 + 1 - 1
\end{align*}
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Let $r \in \mathbb{R}$.

Consider the series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

This is called a geometric series of radius $r$.

Problem: For which $r$ does the geometric series

$$\sum_{n=0}^{\infty} r^n$$

series converge?

To answer this question, we must look at its sequence partial sums;

$$S_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + \cdots + r^k.$$
Geometric Series

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Problem: For which $r$ does the geometric series $\sum_{n=0}^{\infty} r^n$ converge? 

To answer this question, we must look at its sequence partial sums; $S_k = \sum_{n=0}^{k} r^n = 1 + r + r^2 + \cdots + r^k$. 

Brian Forrest
Geometric Series

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Case 1:
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Case 1: $r = 1$
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$$S_k = 1 + 1 + 1 + \cdots + 1$$
Geometric Series

**Problem:** For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

**Case 1: $r = 1$**

$$S_k = 1 + 1 + 1 + \cdots + 1 = k + 1.$$
Geometric Series

**Problem:** For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

**Case 1:** $r = 1$

$$S_k = 1 + 1 + 1 + \cdots + 1 = k + 1.$$ 

Since $\{S_k\} = \{k + 1\}$ diverges,
Geometric Series

**Problem:** For which \( r \) does \( \sum_{n=0}^{\infty} r^n \) series converge?

**Case 1:** \( r = 1 \)

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S_k = 1 + 1 + 1 + \cdots + 1 = k + 1.
\]

Since \( \{S_k\} = \{k + 1\} \) diverges, the series \( \sum_{n=0}^{\infty} 1^n \) diverges.
Problem: For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 2:
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Case 2: \( r = -1 \)
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Case 2: \( r = -1 \)

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S_k = 1 + (-1) + 1 + \cdots + (-1)^k =
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Problem: For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 2: $r = -1$

$$S_k = 1 + (-1) + 1 + \cdots + (-1)^k = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ \end{cases}$$
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Problem: For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 2: $r = -1$

$$S_k = 1 + (-1) + 1 + \cdots + (-1)^k = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Since $\{S_k\} = \{(-1)^k + 1\}$ diverges, $\sum_{n=0}^{\infty} (-1)^n$ diverges.
Geometric Series

**Problem:** For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

**Case 3:**
Geometric Series

**Problem:** For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

**Case 3:** $r \neq 1$
Geometric Series

Problem: For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

Case 3: $r \neq 1$

$$S_k = 1 + r + r^2 + \cdots + r^k$$
Problem: For which \( r \) does \( \sum_{n=0}^{\infty} r^n \) series converge?

Case 3: \( r \neq 1 \)

\[
S_k = 1 + r + r^2 + \cdots + r^k
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\[
rS_k
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Problem: For which \( r \) does \( \sum_{n=0}^{\infty} r^n \) series converge?

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S_k = 1 + r + r^2 + \cdots + r^k
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rS_k = r + r^2 + \cdots + r^k + r^{k+1}
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**Problem:** For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

**Case 3:** $r \neq 1$

$$S_k = 1 + r + r^2 + \cdots + r^k$$

$$rS_k = r + r^2 + \cdots + r^k + r^{k+1}$$

$\Rightarrow$
Problem: For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

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$$S_k = 1 + r + r^2 + \cdots + r^k$$

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$$\Rightarrow (1 - r)S_k$$
**Problem:** For which \( r \) does \( \sum_{n=0}^{\infty} r^n \) series converge?

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S_k = 1 + r + r^2 + \cdots + r^k
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\[
S_k = \frac{1 - r^{k+1}}{1 - r}
\]
Problem: For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

Now

$| r^{k+1} | \rightarrow$
**Problem:** For which $r$ does $\sum_{n=0}^{\infty} r^n$ series converge?

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Geometric Series

Theorem: [Geometric Series Test]

\[ \sum_{n=0}^{\infty} r^n \text{ converges if and only if } |r| < 1. \]

Moreover, if \(|r| < 1\),

\[ \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \]

Example:

\[ \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2 \]

\[ = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots \]

\[ = (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots) - 1 = 1 \]
Theorem: [Geometric Series Test]

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Example:

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\[
\Rightarrow \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\right) - 1 = 2 - 1 = 1
\]
Theorem: [Geometric Series Test]
A geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$. 

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**Example:**

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\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}}
\]
Geometric Series

Theorem: [Geometric Series Test]
A geometric series \( \sum_{n=0}^{\infty} r^n \) converges if and only if \( |r| < 1 \).

Moreover, if \( |r| < 1 \),

\[
\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}
\]

Example:

\[
\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2
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\(\Rightarrow \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots\)
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\Rightarrow \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots)
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\]
Resolving The Paradox of Achilles:

We will call each time Achilles moves from a point to where the tortoise was a stage. 

\[ d_1 \Rightarrow t_1 \quad d_2 \Rightarrow t_2 \quad \cdots \quad d_n \Rightarrow t_n \]

Time to catch the Tortoise: 

\[ T_0 = t_1 + t_2 + \cdots + t_n + \cdots \]

\[ = \sum_{n \to \infty} t_n = T_0 \]

Conclusion: Achilles catches the tortoise at time \( T_0 \)!
Resolving the Paradox of Achilles:

We will call each time Achilles moves from a point to where the tortoise was a stage. Let $d_n = \text{distance traveled in stage } n$, $t_n = \text{time to complete stage } n$. The time to catch the tortoise is:

$$T_0 = t_1 + t_2 + \cdots + t_n + \cdots$$

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$\sum_{n=1}^{\infty} t_n = T_0$

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Conclusion: Achilles catches the tortoise at time \( T_0 \).
Resolving The Paradox of Achilles:

We will call each time Achilles moves from a point to where the tortoise was a stage.

- $d_n = \text{distance traveled in stage } n \Rightarrow t_n = \text{time to complete stage } n$

Resolving the Paradox:

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Time to catch the Tortoise
Resolving The Paradox of Achilles:

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We will call each time Achilles moves from a point to where the tortoise was a stage.

$\bullet$ $d_n =$ distance traveled in stage $n \Rightarrow t_n =$ time to complete stage $n$

Time to catch the Tortoise $=$
Resolving The Paradox of Achilles:

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We will call each time Achilles moves from a point to where the tortoise was a stage.

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Time to catch the Tortoise \( = t_1 \)
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Resolving the Paradox:
We will call each time Achilles moves from a point to where the tortoise was a stage.

\[ d_n = \text{distance traveled in stage } n \implies t_n = \text{time to complete stage } n \]

Time to catch the Tortoise \( = t_1 + t_2 \)
Resolving The Paradox of Achilles:

We will call each time Achilles moves from a point to where the tortoise was a *stage*.

- \( d_n = \text{distance traveled in stage } n \Rightarrow t_n = \text{time to complete stage } n \)

Time to catch the Tortoise \( = t_1 + t_2 + \cdots + t_n \)
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**Conclusion:** Achilles catches the tortoise at time $T_0$!