Limits of Sequences

Brian Forrest

October 1, 2010
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Consider \( \left\{ \frac{1}{n} \right\} \). As \( n \) gets larger and larger, the terms get closer and closer to \( r_p \). We want to call \( r_n \) the limit of the sequence \( \left\{ \frac{1}{n} \right\} \) as \( n \) goes to \( \infty \).
Consider \( \left\{ \frac{1}{n} \right\} \). As \( n \) gets larger and larger, the terms get closer and closer to \( r \). We want to call \( r \) the limit of the sequence \( \left\{ \frac{1}{n} \right\} \) as \( n \) goes to \( \infty \).
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We want to call 0, the limit of the sequence $\left\{ \frac{1}{n} \right\}$ as $n$ goes to $\infty$. 
Heuristic Definition:

We say the limit of the sequence \( \{a_n\} \) as \( n \to \infty \) if as \( n \) gets larger and larger, the terms of \( \{a_n\} \) get closer and closer to \( L \).

Question: What's wrong with this definition?
Heuristic Definition: We say the \( L \) is the limit of the sequence \( \{a_n\} \) as \( n \) goes to \( \infty \).
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Heuristic Definition: We say the $L$ is the *limit of the sequence* \( \{a_n\} \) as $n$ goes to $\infty$ if as $n$ gets larger and larger the terms of \( \{a_n\} \) get closer and closer to $L$.

Question: What’s wrong with this definition?
Again, consider \( \left\{ \frac{1}{n} \right\} \).
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Again, consider \( \{ \frac{1}{n} \} \). As \( n \) gets larger and larger
Again, consider \( \left\{ \frac{1}{n} \right\} \). As \( n \) gets larger and larger the terms get closer and closer to 0.
Again, consider \( \left\{ \frac{1}{n} \right\} \). As \( n \) gets larger and larger the terms get closer and closer to 0. But they also get closer and closer to \(-1\).
Question:

What is it about \( r \) that makes us call it the limit of \( \{ \frac{1}{n} \} \) but we do not call \( -s \) the limit as well?
Question: What is it about 0 that makes us call it the limit of \( \left\{ \frac{1}{n} \right\} \) but we do not call \(-1\) the limit as well?
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Answer:
**Question:** What is it about 0 that makes us call it the limit of \( \{\frac{1}{n}\} \) but we do not call \(-1\) the limit as well?

**Answer:** The terms of \( \{\frac{1}{n}\} \) approximate 0 as **closely as we would like** when \( n \) is large enough,
Question: What is it about 0 that makes us call it the limit of \( \{ \frac{1}{n} \} \) but we do not call \(-1\) the limit as well?

Answer: The terms of \( \{ \frac{1}{n} \} \) approximate 0 as closely as we would like when \( n \) is large enough, but they never even get within 1 unit of \(-1\).
Definition: *(New Heuristic Definition)*

We say that \( L \) is the limit of the sequence \( \{a_n\} \) as \( n \) goes to infinity if no matter what positive tolerance \( \epsilon > 0 \) we are given we can find a cutoff \( N \in \mathbb{N} \) such that the terms \( a_n \) approximate \( L \) with error less than \( \epsilon \) provided that \( n \geq N \).

**Formal Definition:**

We say that \( L \) is the limit of the sequence \( \{a_n\} \) as \( n \) goes to infinity if for every \( \epsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that if \( n \geq N \) then \( |a_n - L| < \epsilon \).

In this case we write \( \lim_{n \to \infty} a_n = L \).
Definition of a limit

**Definition: (New Heuristic Definition)**

We say that \( L \) is the *limit* of the sequence \( \{a_n\} \) as \( n \) goes to infinity, if no matter what positive tolerance \( \epsilon \) we are given we can find a cutoff \( N \in \mathbb{N} \) such that the terms \( a_n \) approximate \( L \) with error less than \( \epsilon \) provided that \( n \geq N \).
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We say that $L$ is the limit of the sequence $\{a_n\}$ as $n$ goes to infinity, if no matter what positive tolerance $\epsilon > 0$ we are given, we can find a cutoff $N \in \mathbb{N}$, such that the terms $a_n$ approximate $L$ with error less than $\epsilon$. 

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$$|a_n - L| < \epsilon.$$ 

In this case we write

$$\lim_{n \to \infty} a_n = L.$$
Definition of a limit:
Definition of a limit:

1. Identify $L$. 

Given a smaller $\epsilon$, repeat 3 with a larger $N$. 

Specify the error $\epsilon$. Find the cutoff $N$.
Definition of a limit:

1. Identify $L$.
2. Specify the error $\epsilon > 0$. 
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4. Given a smaller \( \epsilon \).
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5. Repeat 3

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Limits of Sequences
Definition of a limit:

1. Identify $L$.
2. Specify the error $\epsilon > 0$.
3. Find the cutoff $N$.
4. Given a smaller $\epsilon$.
5. **Repeat 3**
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Definition of a limit:

It is useful to look at how this works on the real line.

\[ \text{L} \]

Assume \( \lim_{n \to \infty} a_n = L \).

We create an error band by moving \( \epsilon \) units to the left from \( L \) to \( L - \epsilon \), and then \( \epsilon \) units to the right from \( L \) to \( L + \epsilon \). This gives us \( L - \epsilon, L + \epsilon \) as the "target".

Not all terms in \( \{a_n\} \) must fall in \( L - \epsilon, L + \epsilon \).

We can find \( N \in \mathbb{N} \) such that \( n \geq N \Rightarrow a_n \in (L - \epsilon, L + \epsilon) \).
Definition of a limit:

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- Not all terms in \( \{a_n\} \) must fall in \((L - \epsilon, L + \epsilon)\).
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- Not all terms in \( \{a_n\} \) must fall in \((L - \epsilon, L + \epsilon)\).
- We can find \( N \in \mathbb{N} \) such that \( n \geq N \Rightarrow a_n \in (L - \epsilon, L + \epsilon) \).
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- Not all terms in $\{a_n\}$ must fall in $(L - \epsilon, L + \epsilon)$.
- We can find $N \in \mathbb{N}$ such that $n \geq N \implies a_n \in (L - \epsilon, L + \epsilon)$
Definition of a limit:

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Assume that
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Choose
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Assume that
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Choose
\[ \epsilon \leq \min\{L - a, b - L\}. \]
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Then
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(L - \epsilon, L + \epsilon) \subseteq (a, b).
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Then
\[ (L - \epsilon, L + \epsilon) \subseteq (a, b). \]

If \( n \) is large enough, then \( a_n \in (L - \epsilon, L + \epsilon) \)
Definition of a limit:

Assume that
\[ \lim_{n \to \infty} a_n = L \in (a, b). \]

Choose
\[ \epsilon \leq \min\{L - a, b - L\}. \]

Then
\[ (L - \epsilon, L + \epsilon) \subseteq (a, b). \]

If \( n \) is large enough, then \( a_n \in (L - \epsilon, L + \epsilon) \) and hence
\[ a_n \in (a, b). \]
Summary:

The following statements can all be viewed as being equivalent:
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1. \( \lim_{n \to \infty} a_n = L. \)
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The following statements can all be viewed as being equivalent:

1. \( \lim_{n \to \infty} a_n = L \).
2. Every interval \((L - \epsilon, L + \epsilon)\) contains a tail of \(\{a_n\}\).

Changing finitely many terms in \(\{a_n\}\) does not affect convergence.
Summary:

The following statements can all be viewed as being equivalent:

1. \( \lim_{n \to \infty} a_n = L. \)
2. Every interval \((L - \epsilon, L + \epsilon)\) contains a tail of \(\{a_n\}\).
3. Every interval \((L - \epsilon, L + \epsilon)\) contains all but finitely many terms of \(\{a_n\}\).

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3. Every interval \((L - \epsilon, L + \epsilon)\) contains all but finitely many terms of \(\{a_n\}\).
4. Every interval \((a, b)\) containing \(L\) contains a tail of \(\{a_n\}\).
Summary:

The following statements can all be viewed as being equivalent:

1. \( \lim_{n \to \infty} a_n = L. \)
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4. Every interval \((a, b)\) containing \(L\) contains a tail of \(\{a_n\}\).
5. Every interval \((a, b)\) containing \(L\) contains all but finitely many terms of \(\{a_n\}\).
6. Changing finitely many terms in \(\{a_n\}\) does not affect convergence.