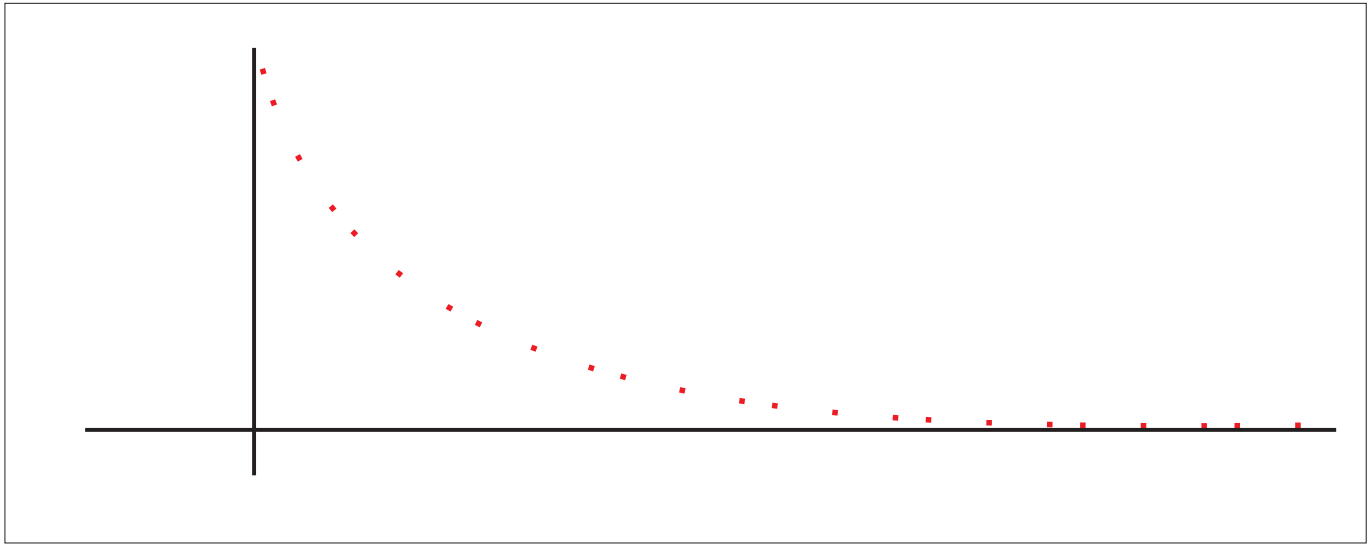
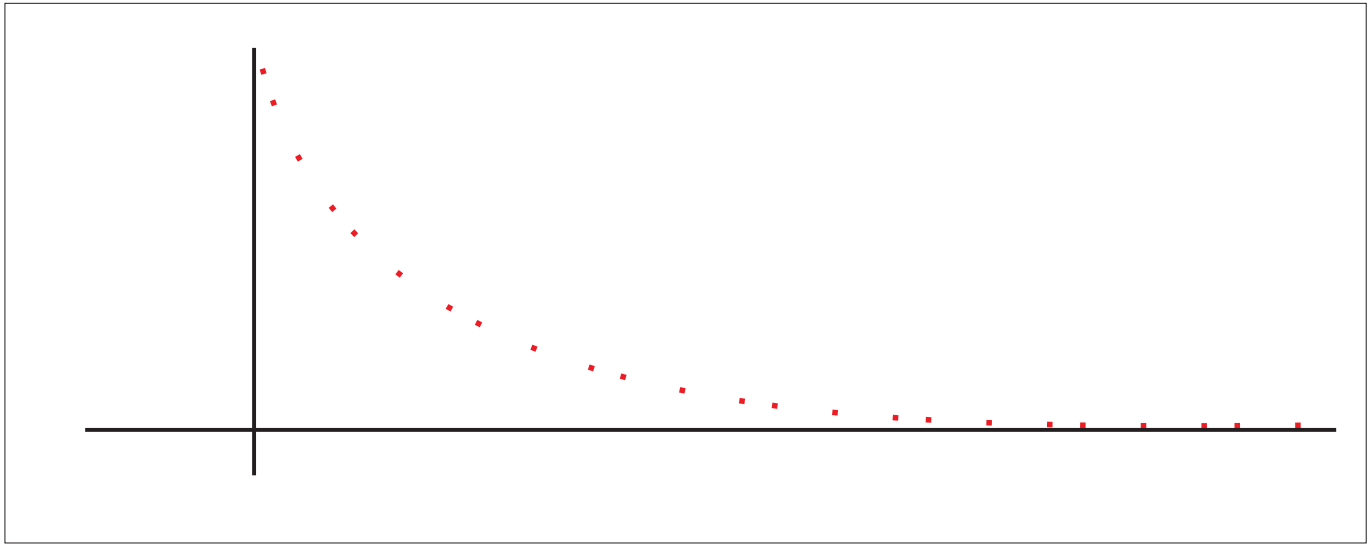
Limits of Sequences

Brian Forrest

October 1, 2010



Consider $\{\frac{1}{n}\}$.



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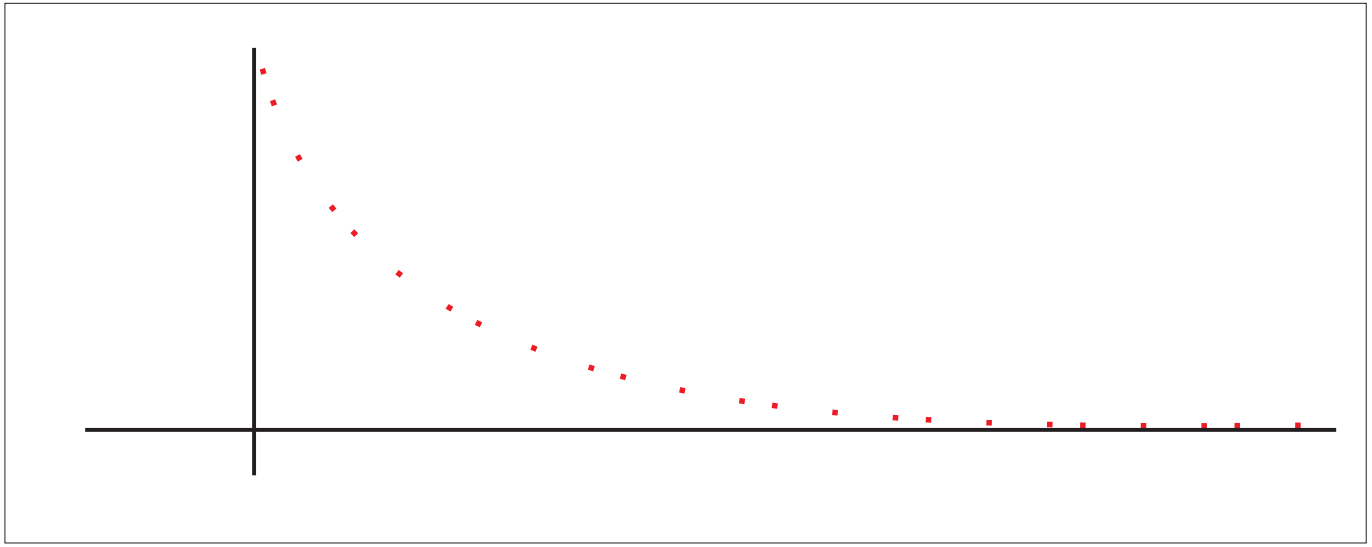

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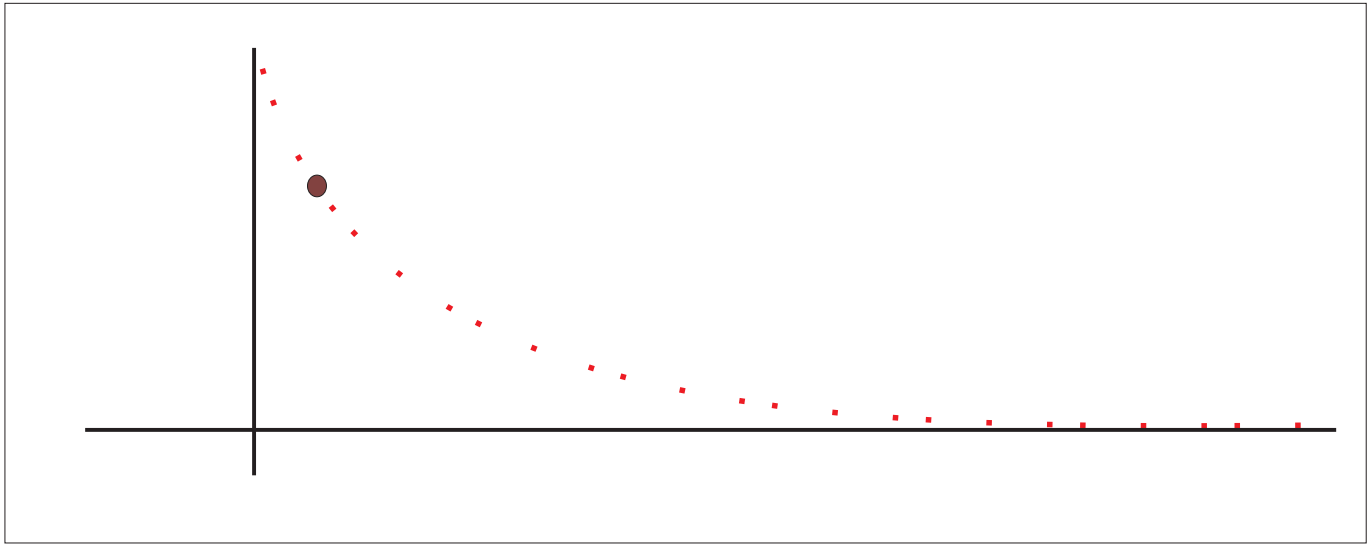

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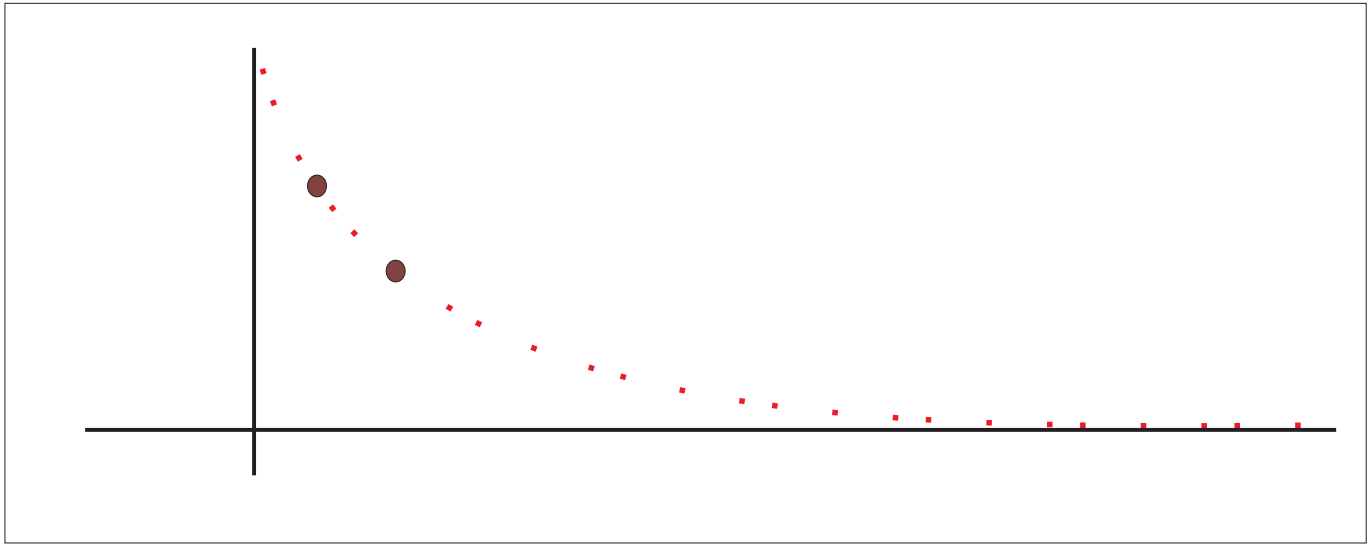

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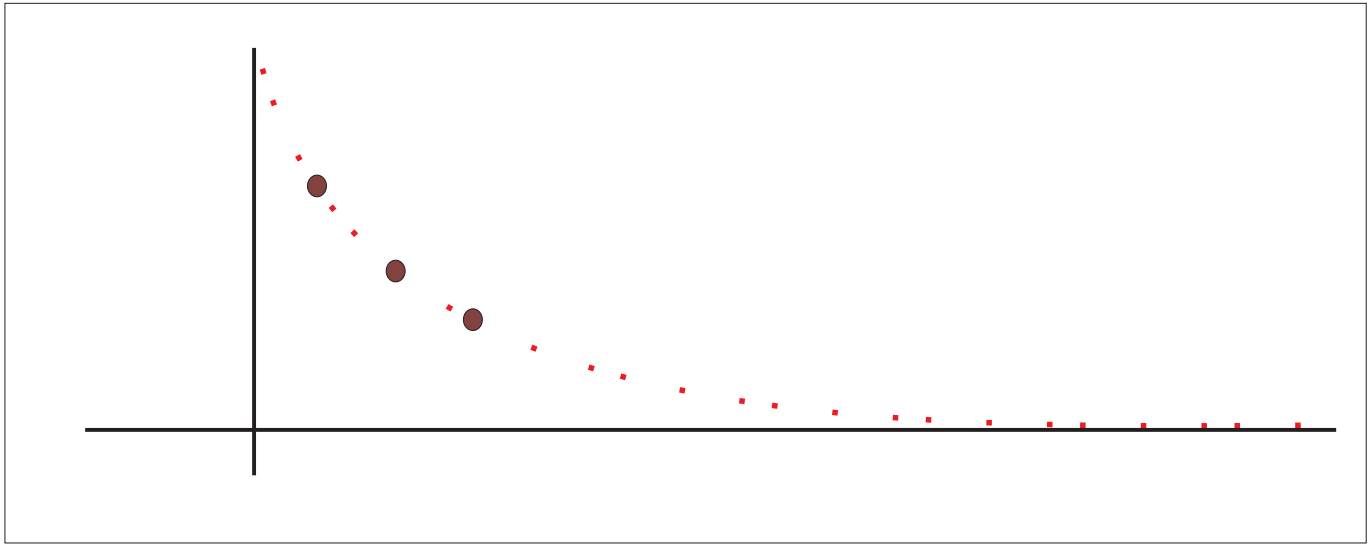

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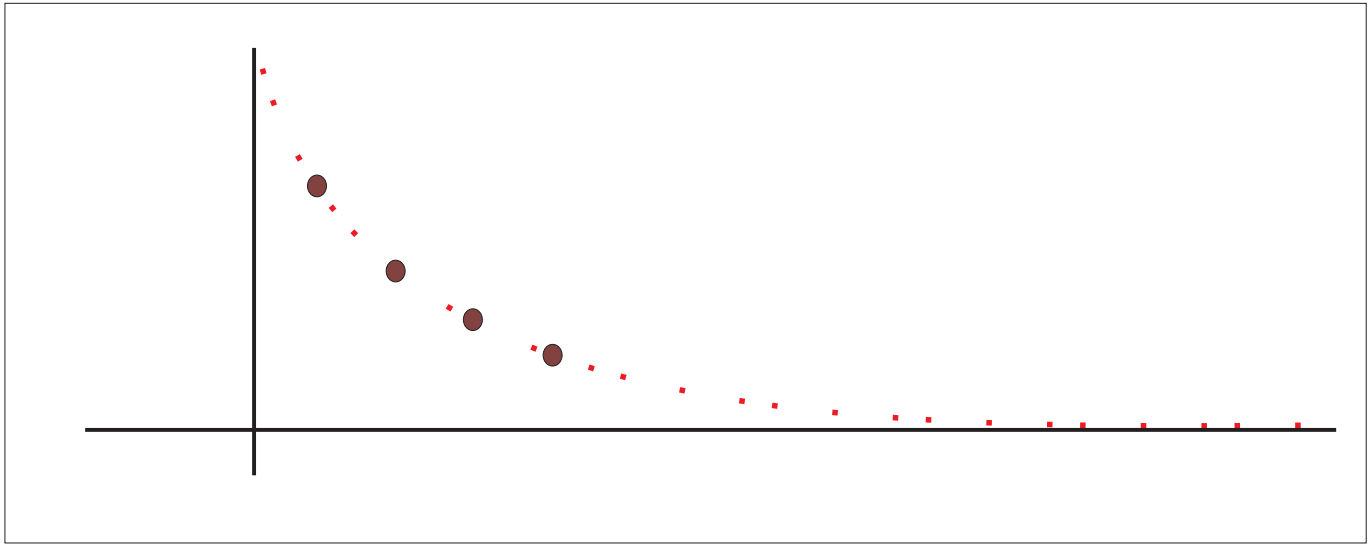

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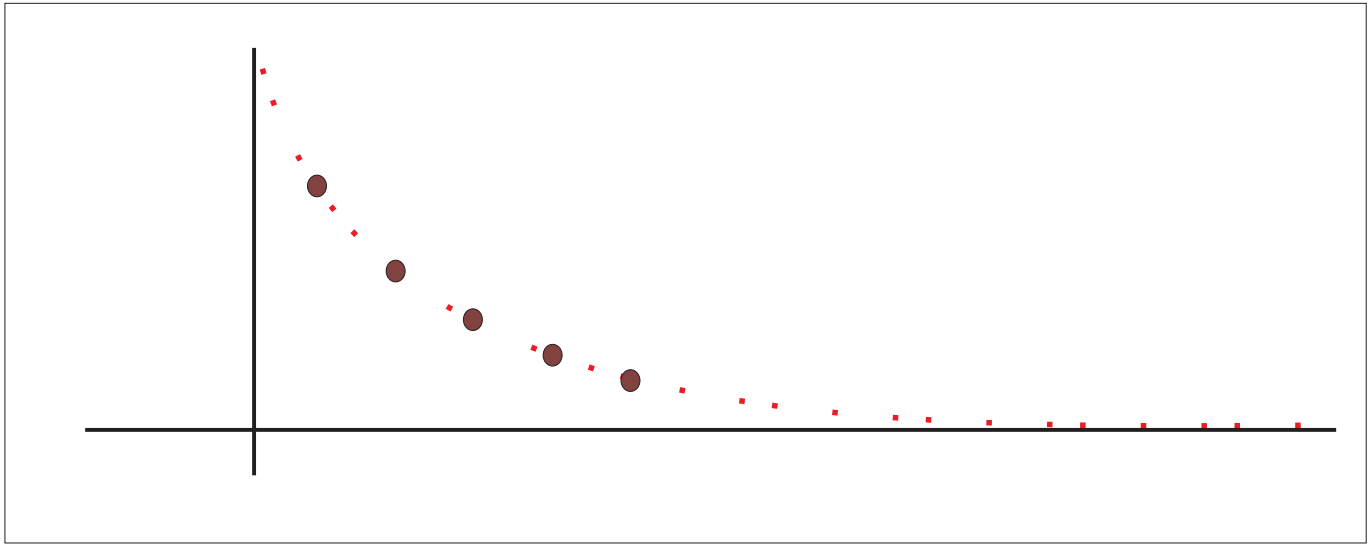

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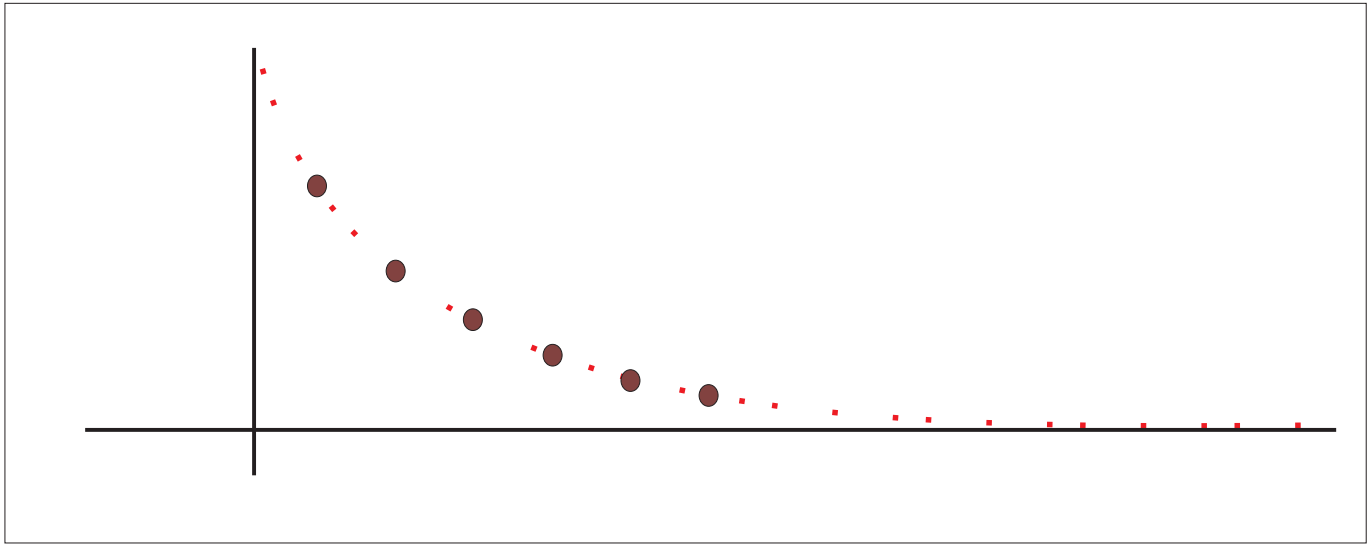

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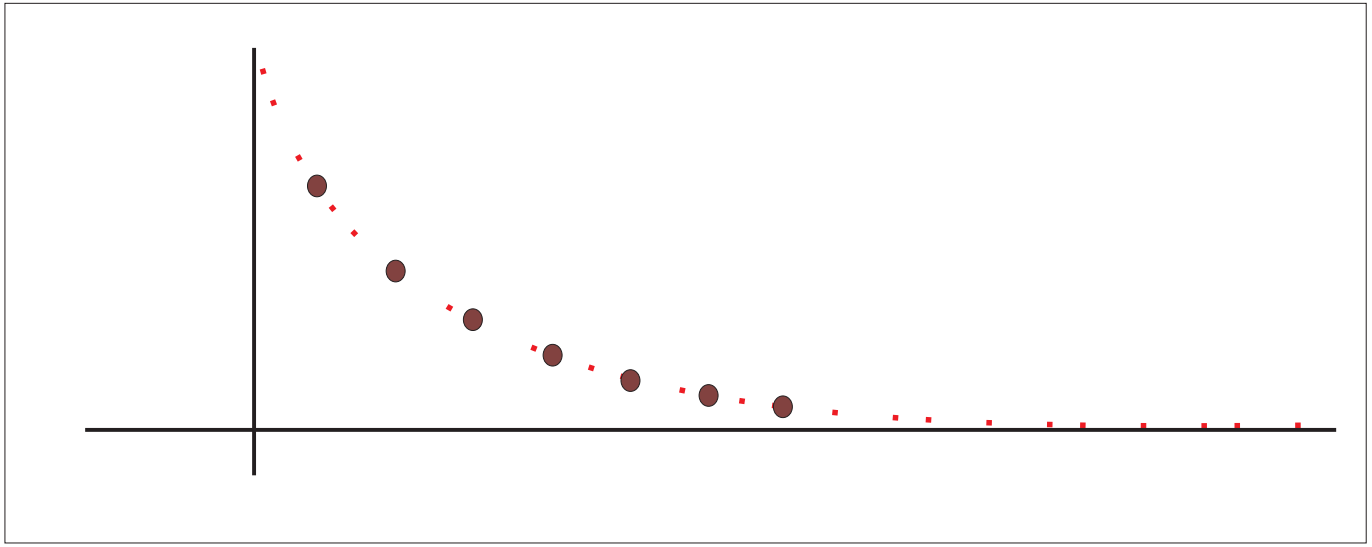

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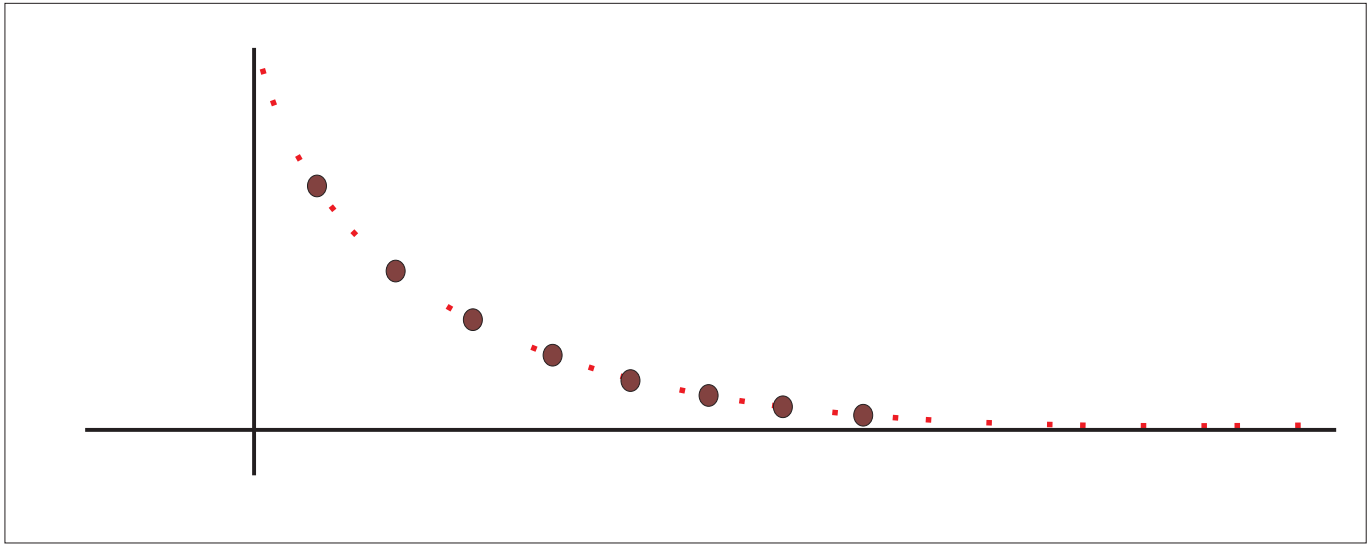

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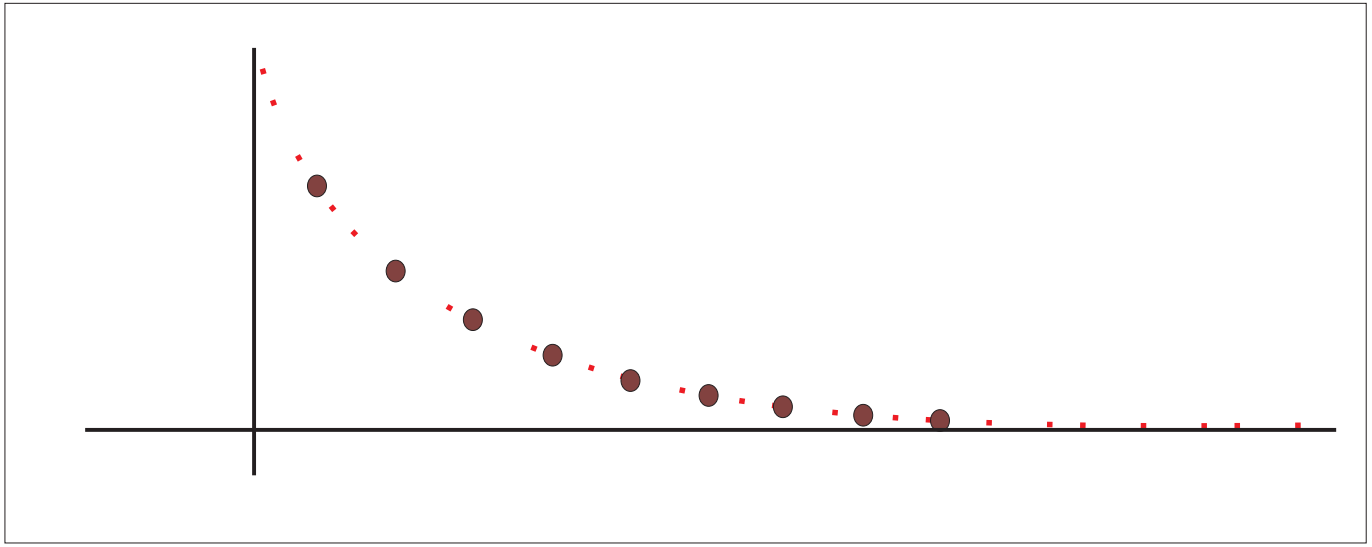

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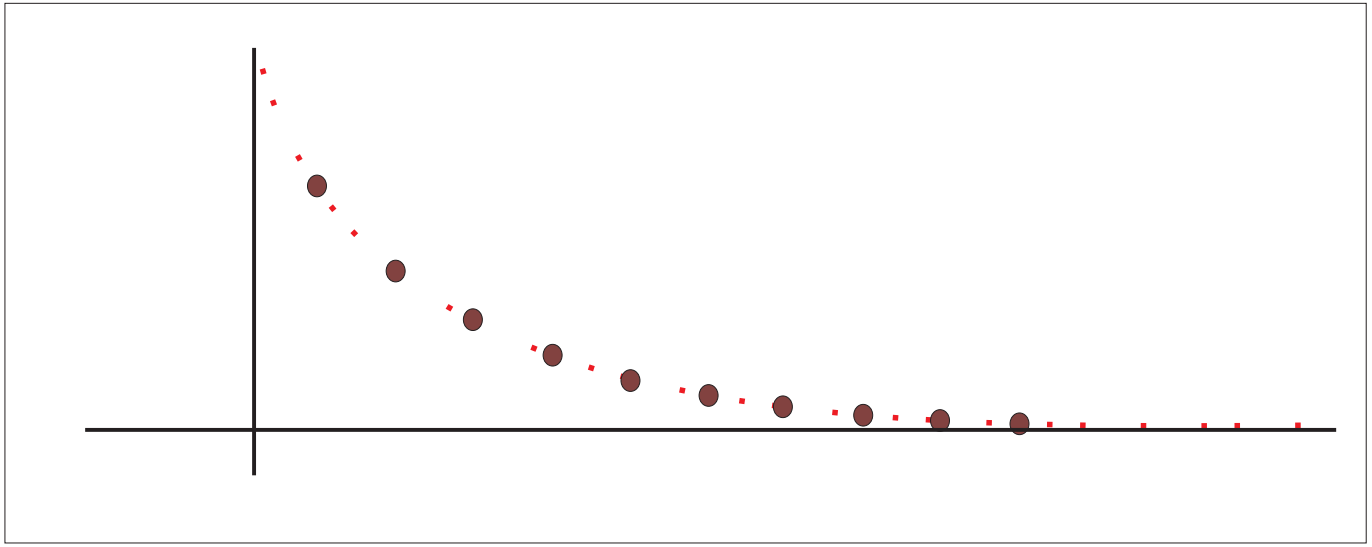

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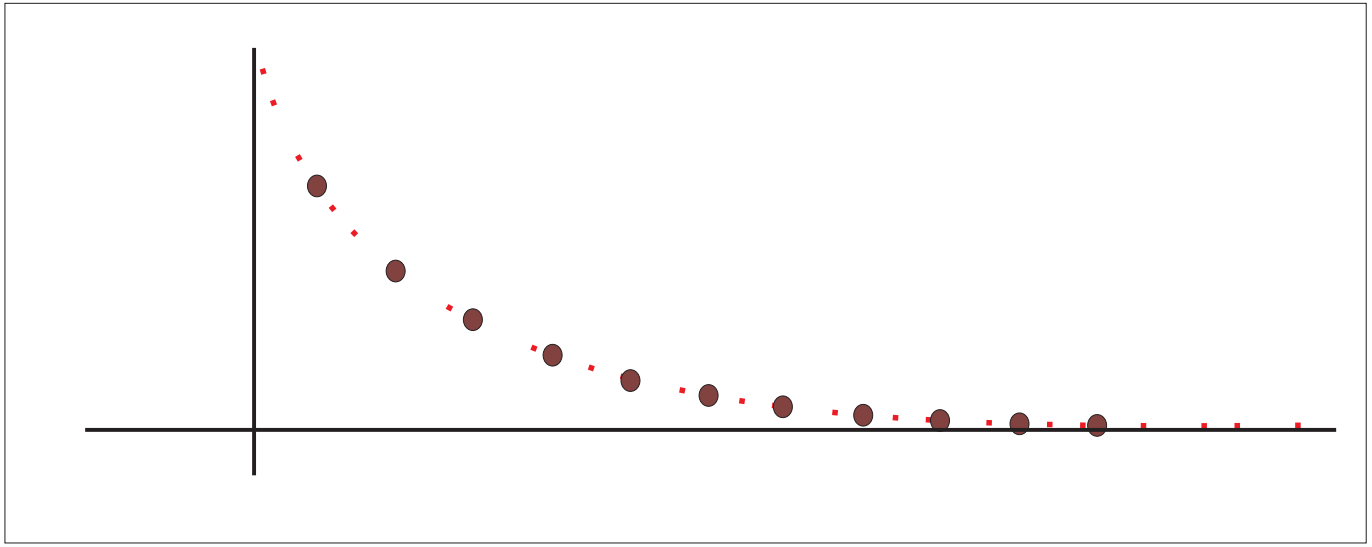

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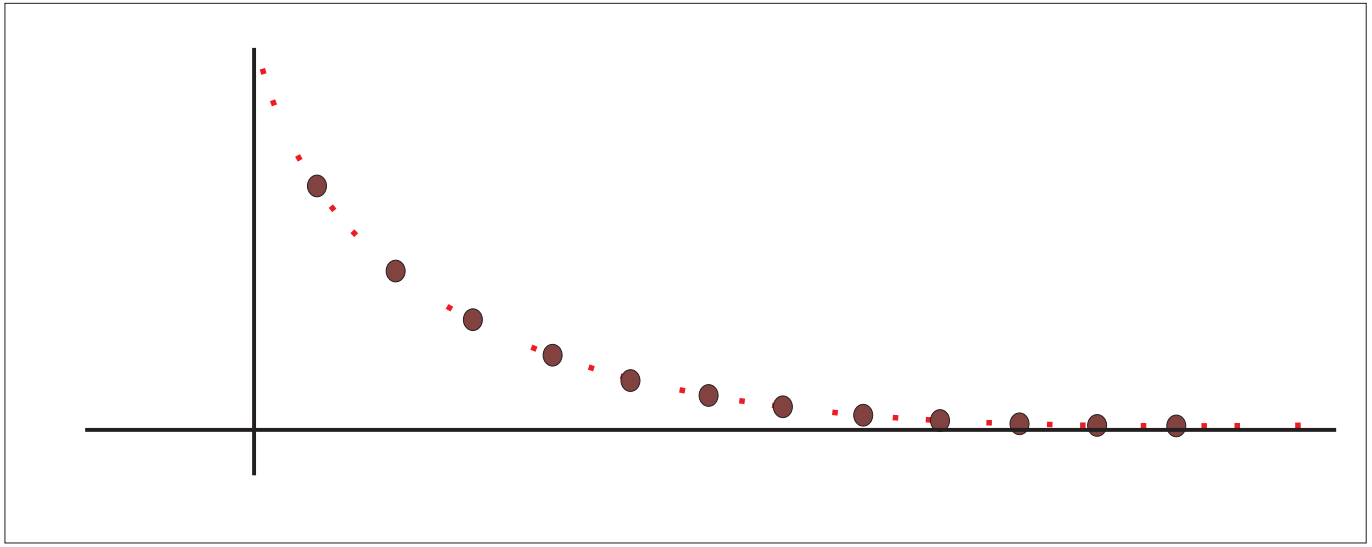

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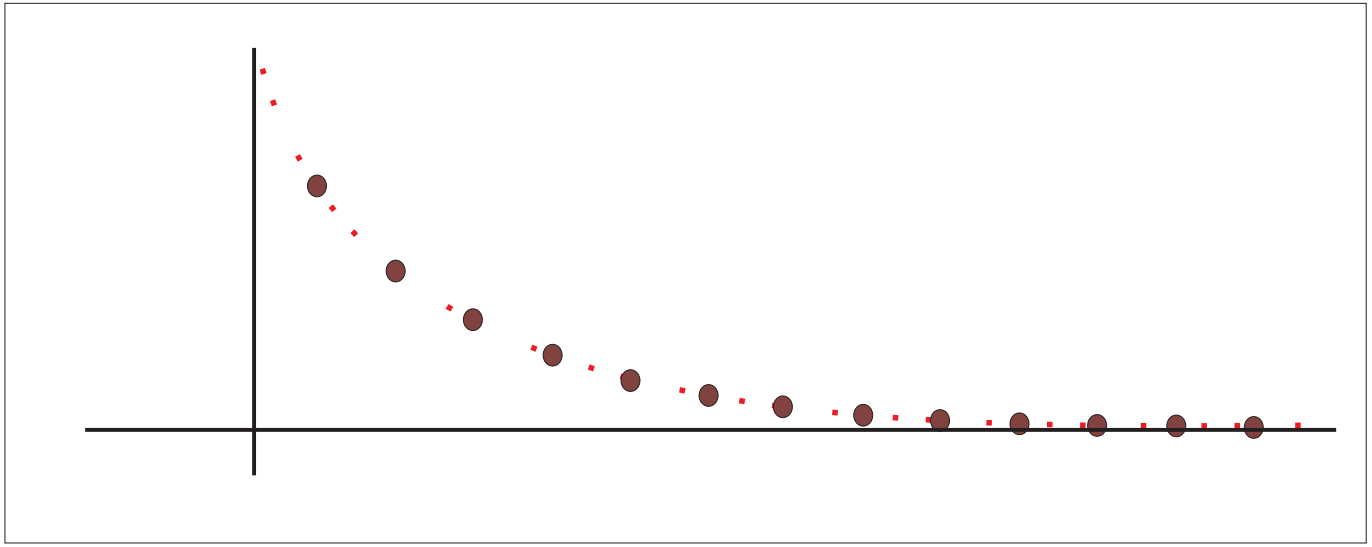

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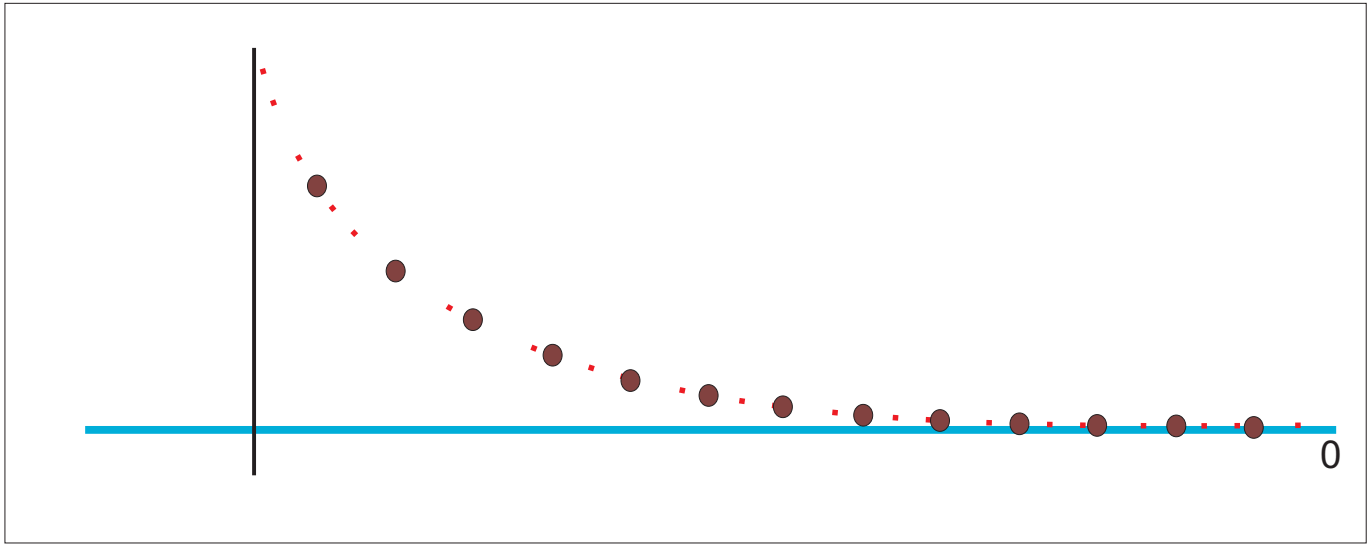

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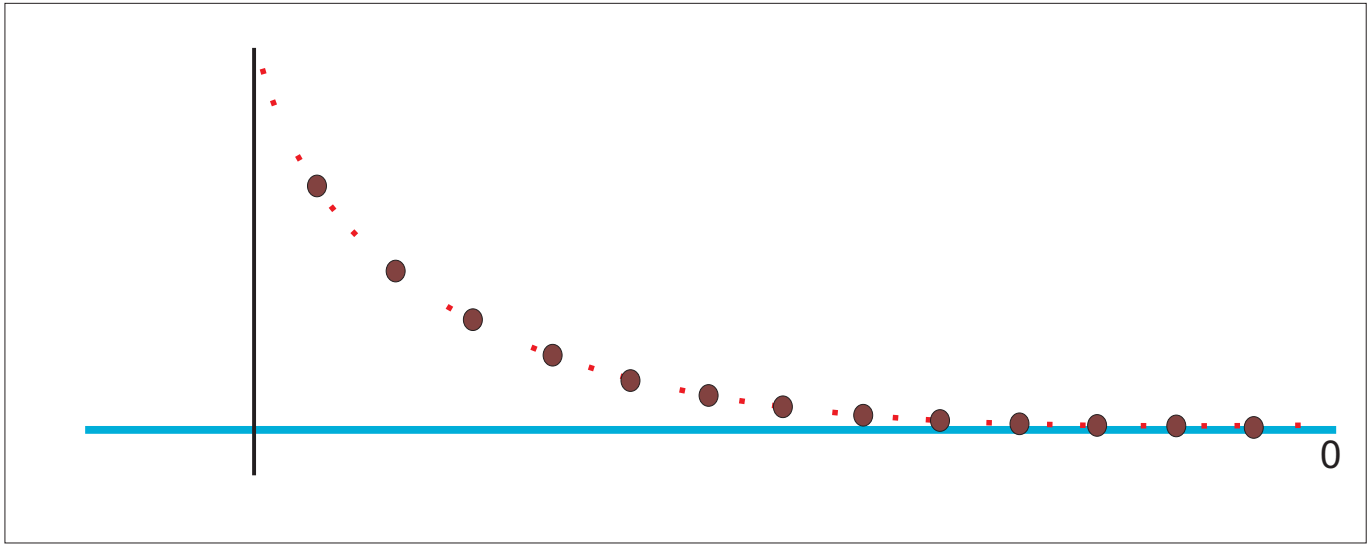
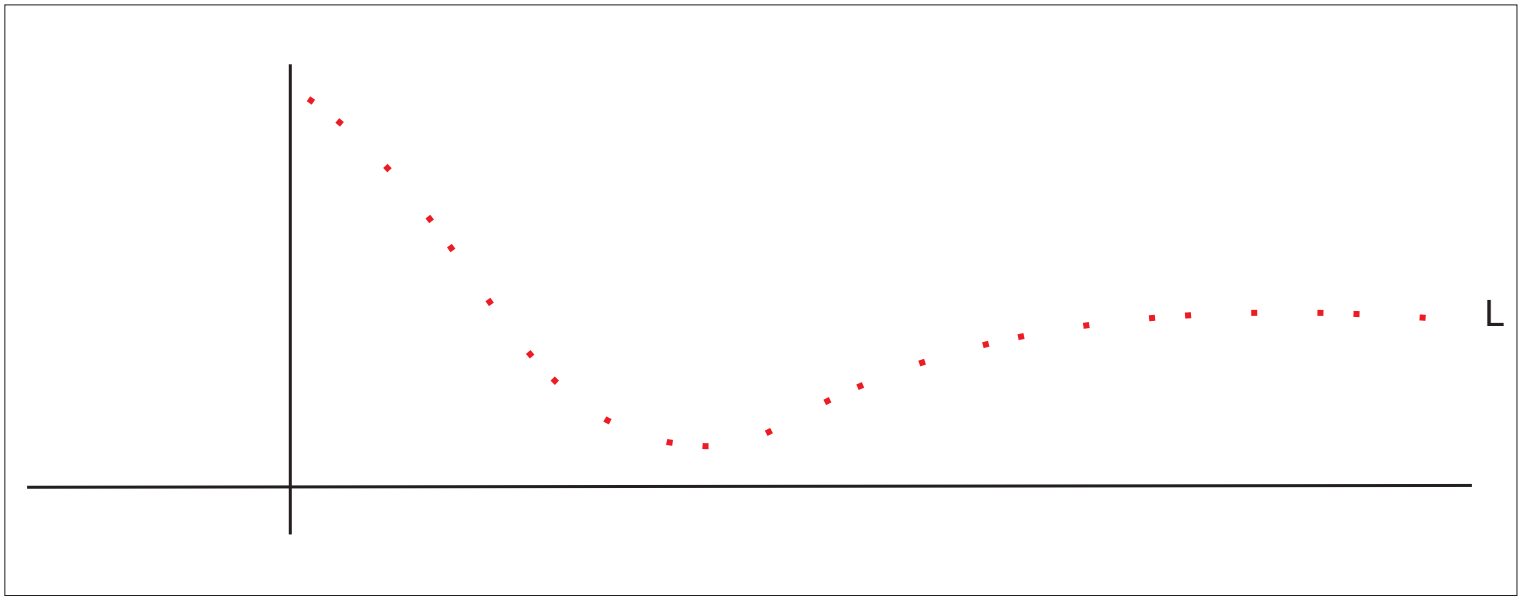


Consider $\{\frac{1}{n}\}$. As n **gets larger and larger** the terms get **closer and closer to** 0.

We want to call 0, *the limit of the sequence* $\{\frac{1}{n}\}$ as n goes to ∞.

Heuristic Definition



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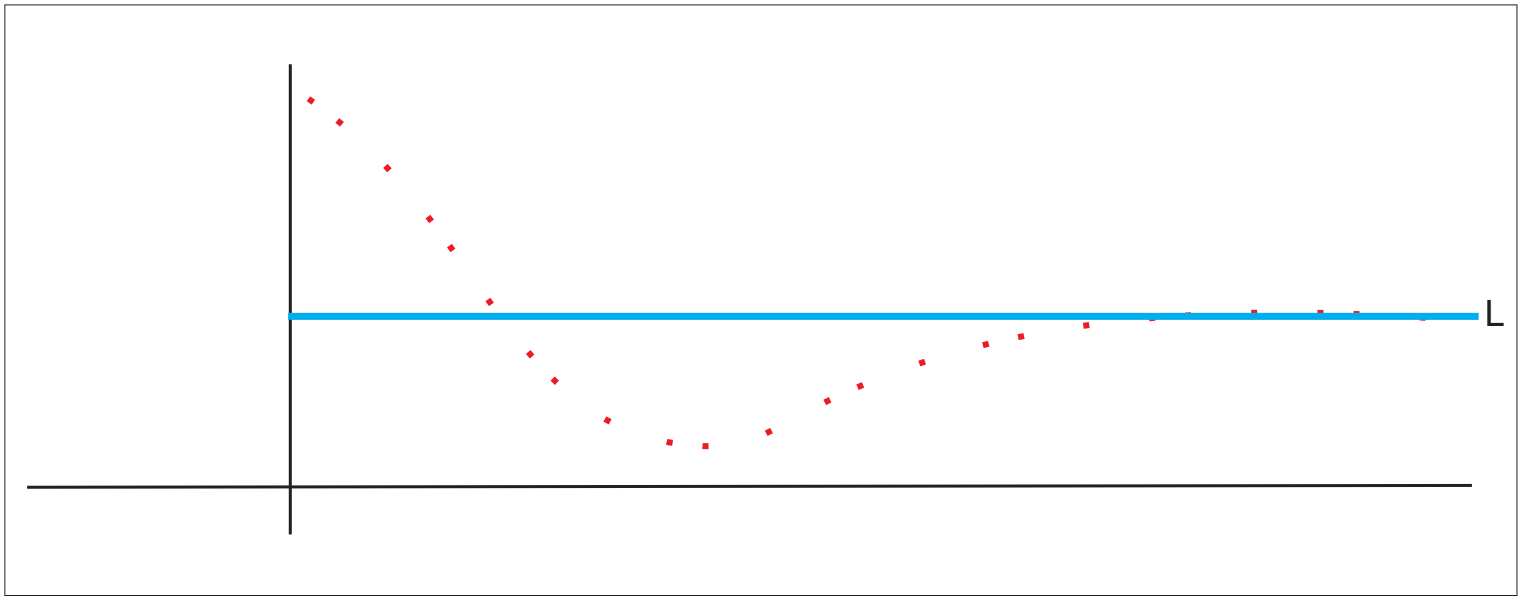

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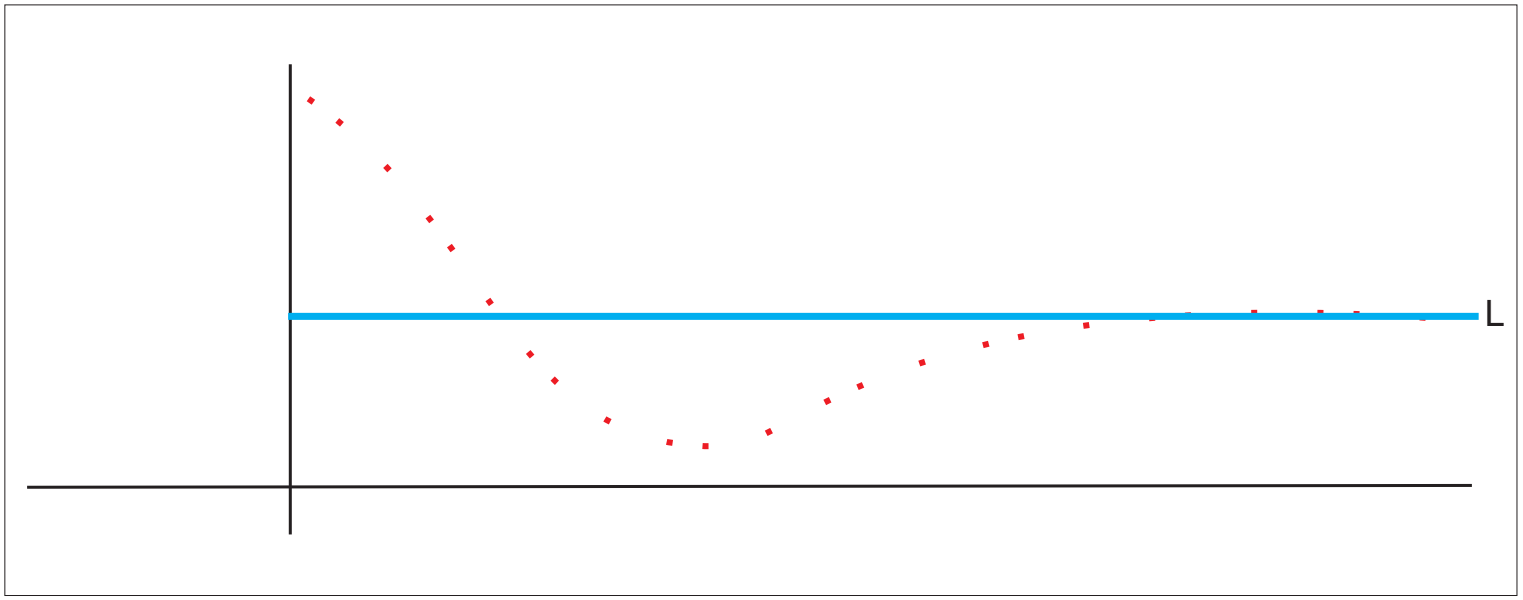

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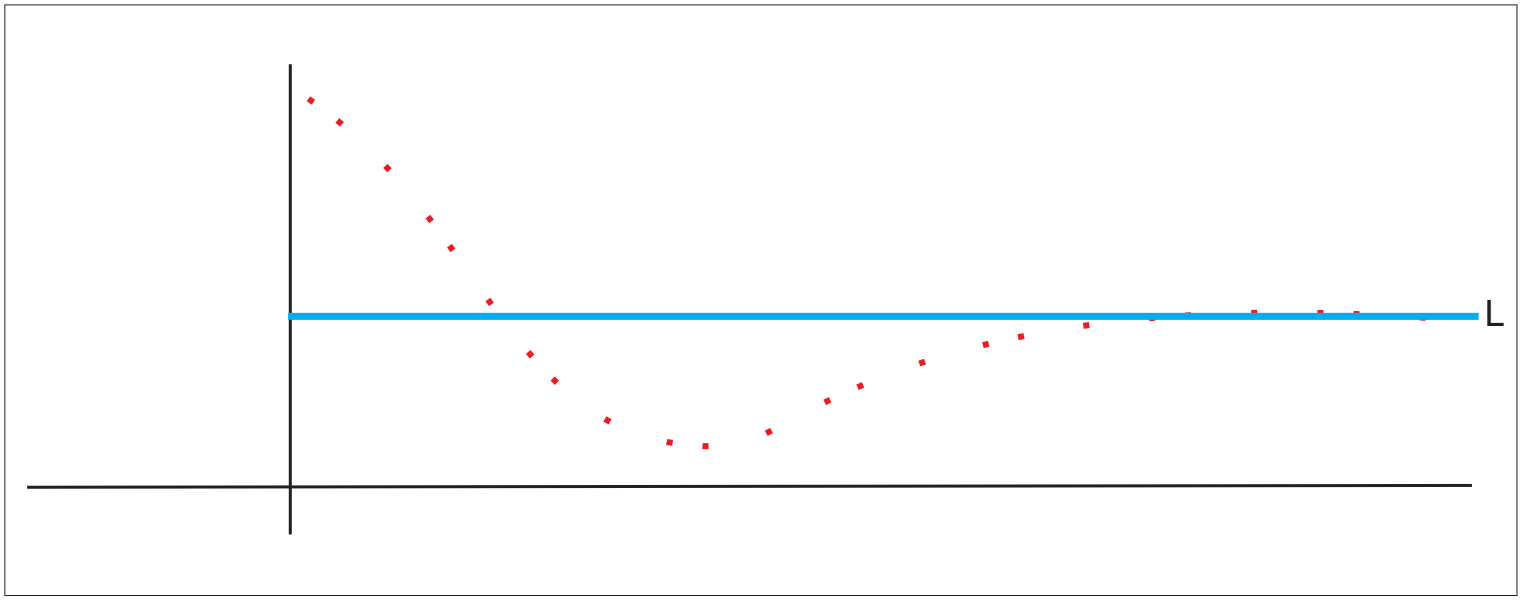

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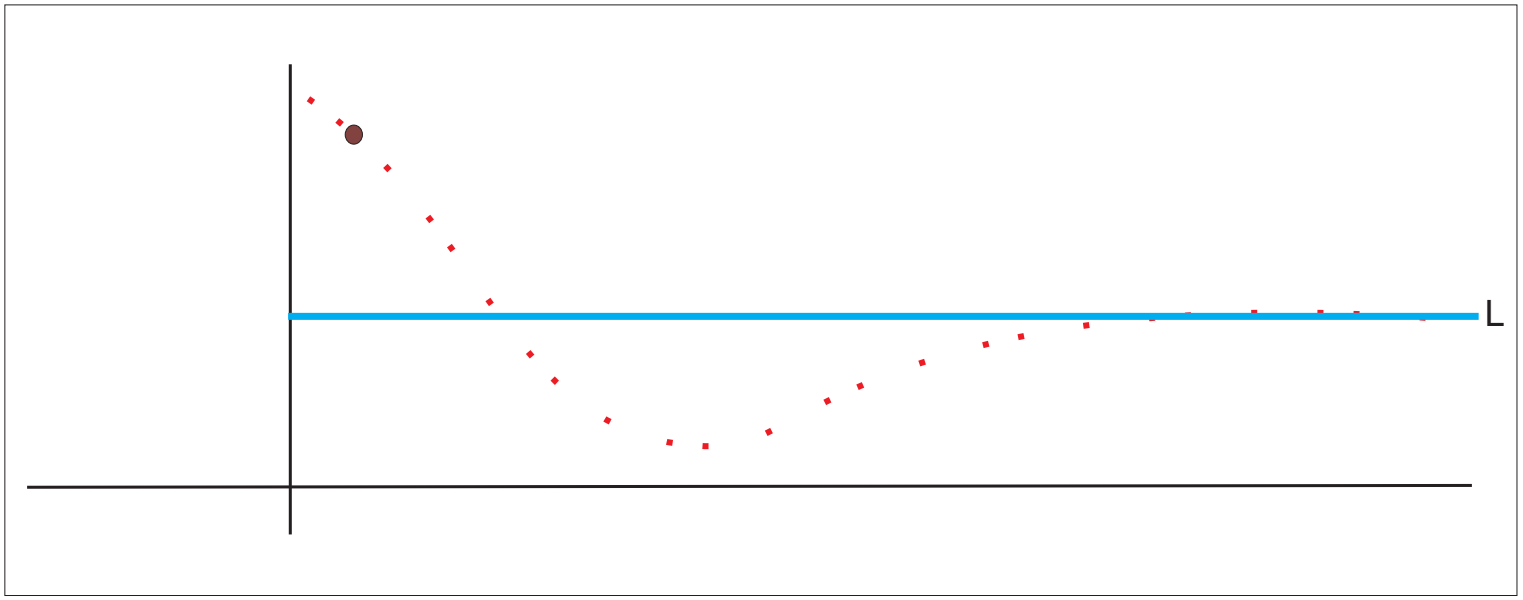

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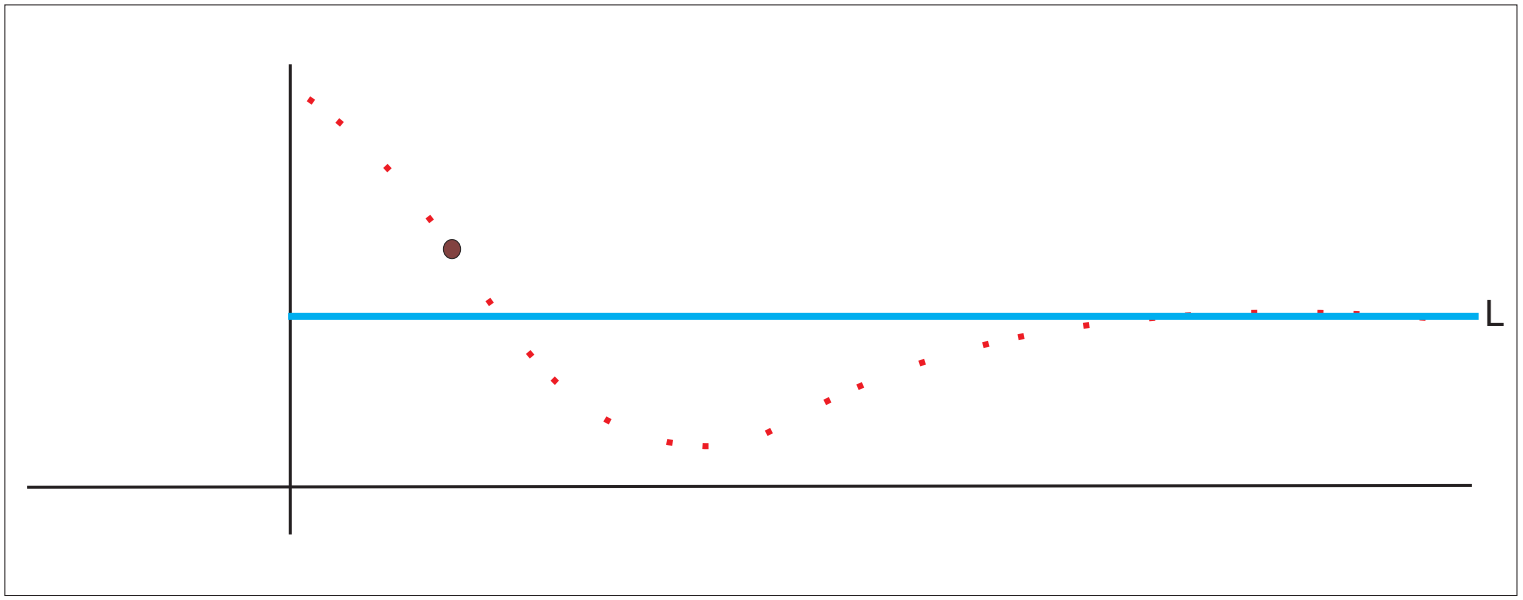

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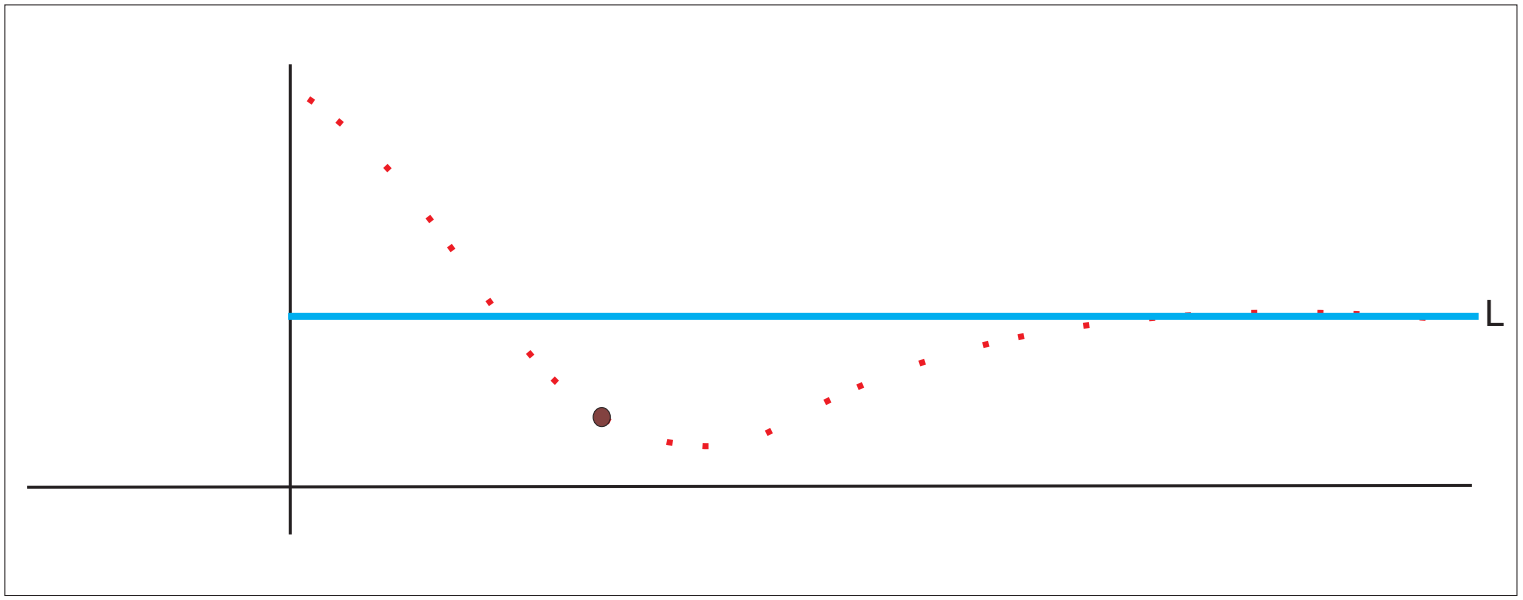

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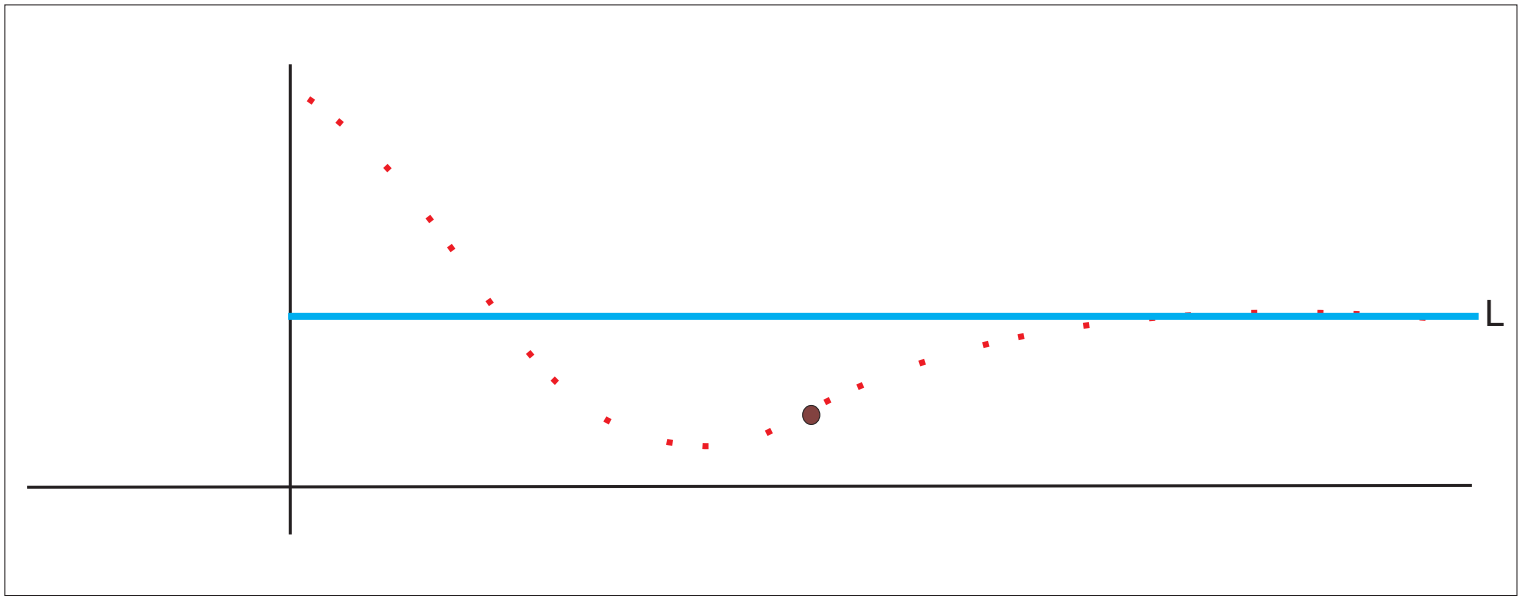

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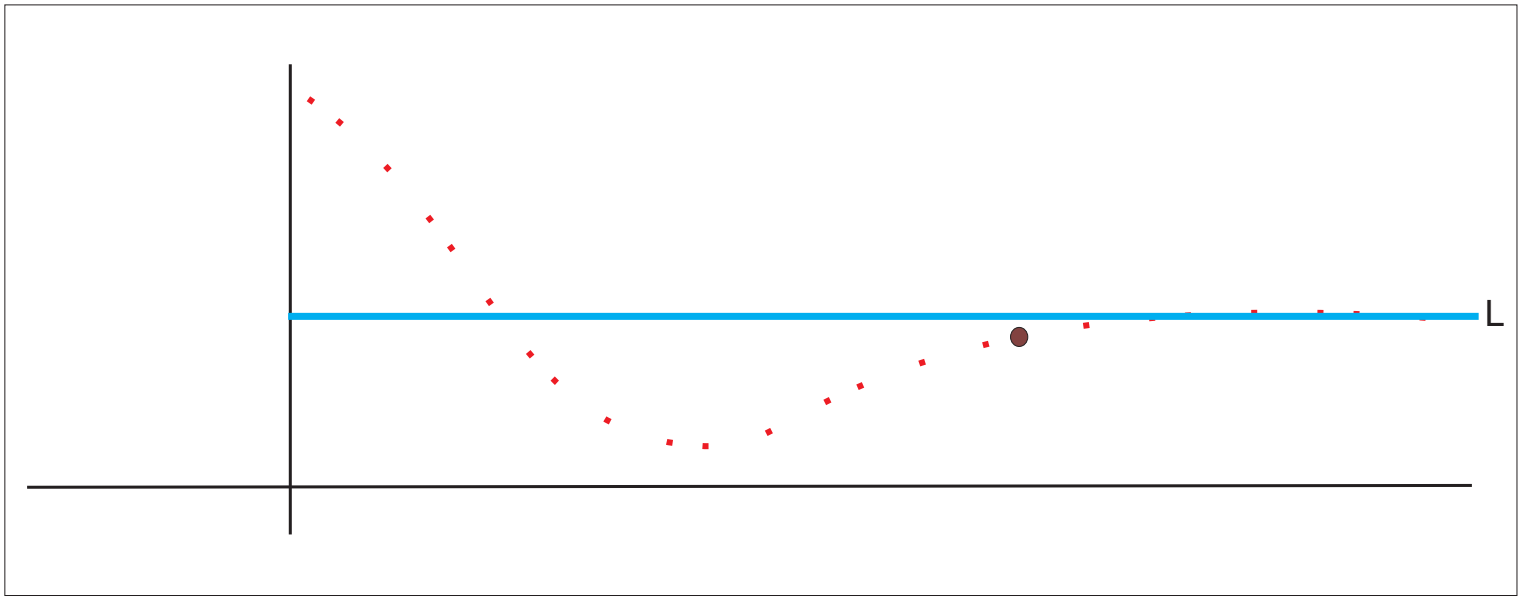

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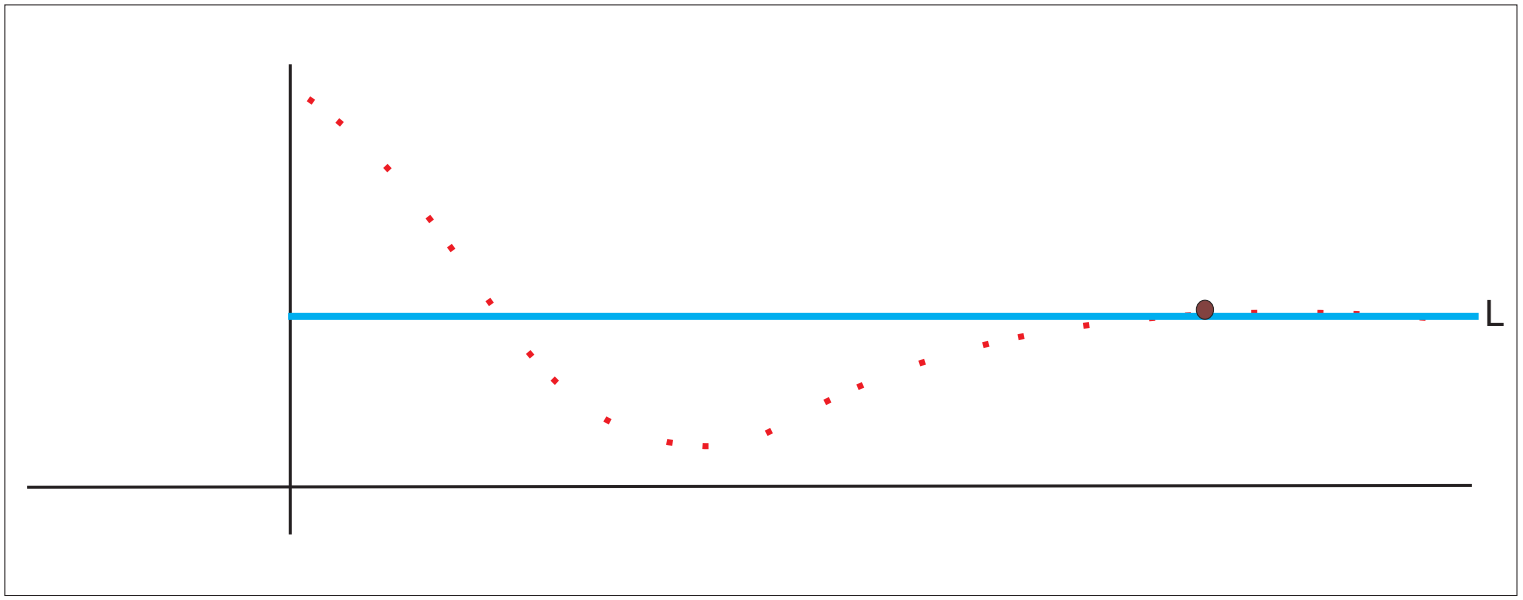

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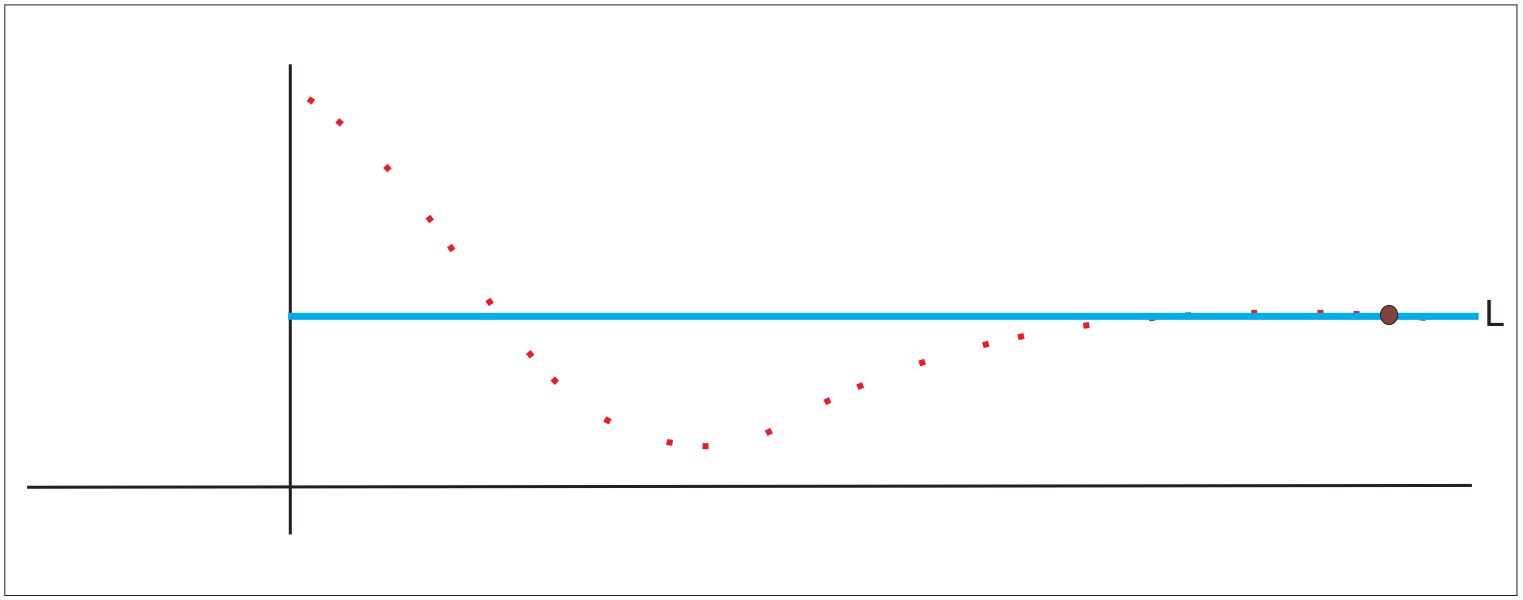

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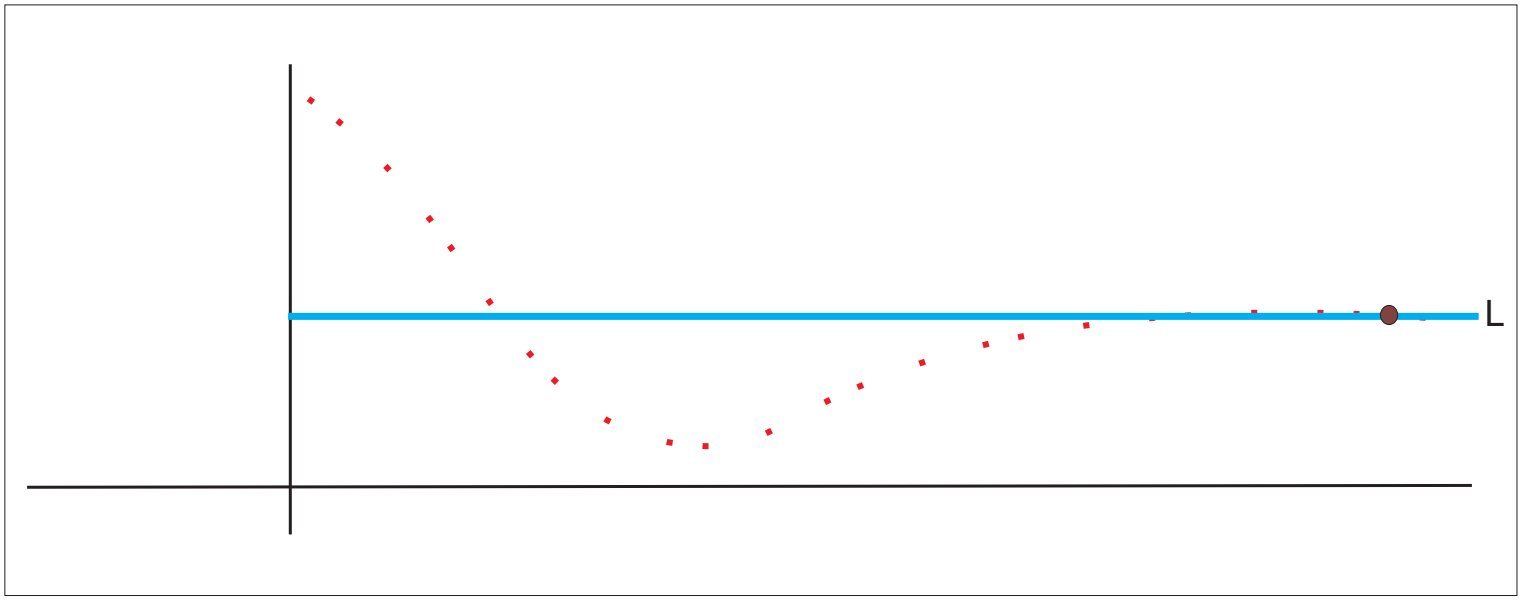

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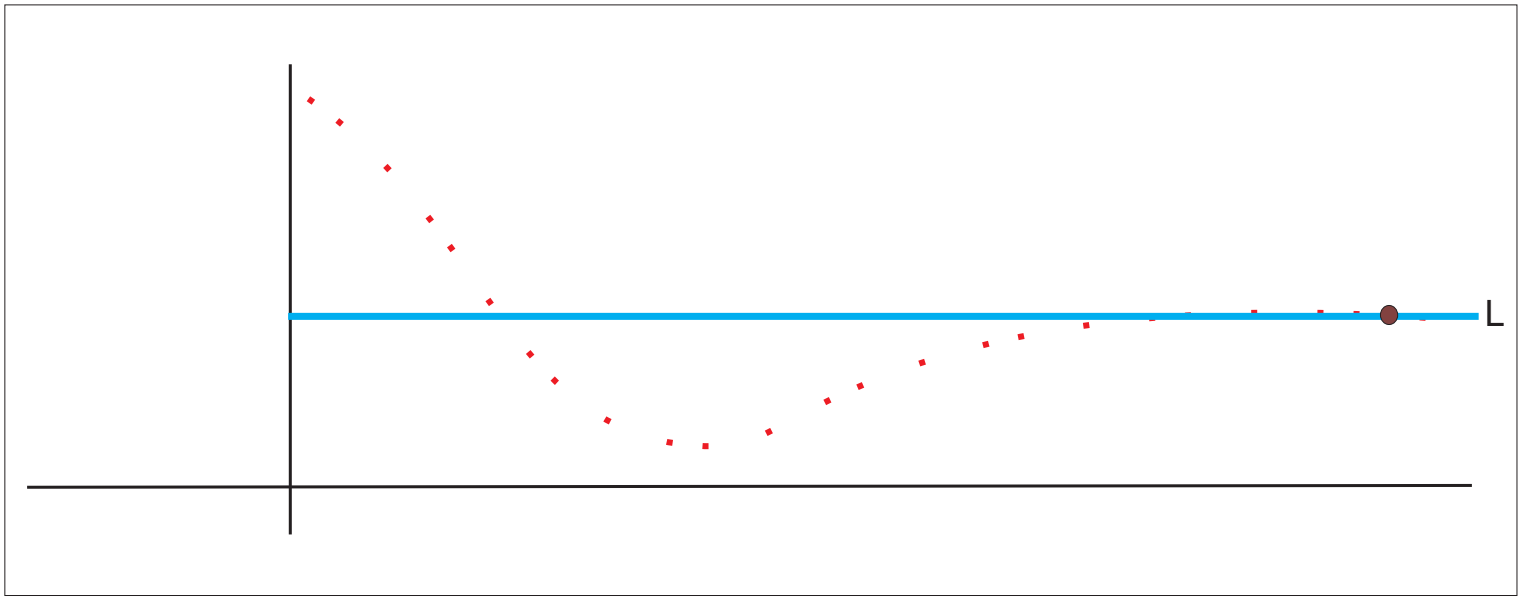
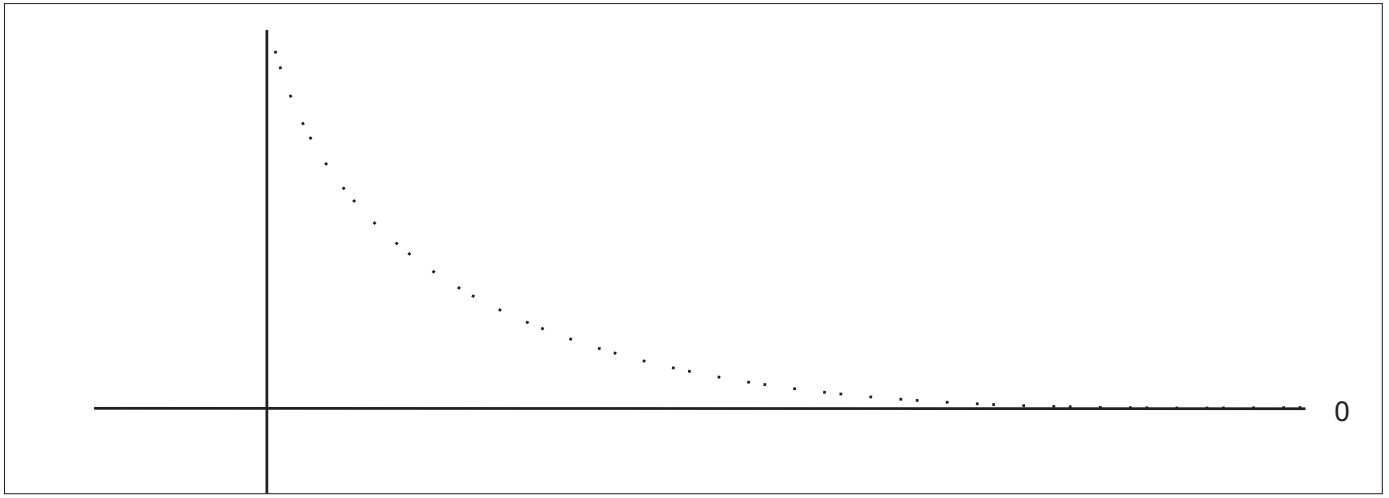


Heuristic Definition: We say the L is the *limit of the sequence* $\{a_n\}$ as n goes to ∞ if as n **gets larger and larger** the terms of $\{a_n\}$ get **closer and closer** to L.

Question: What's wrong with this definition?

Heuristic Definition



Again, consider $\{\frac{1}{n}\}$.

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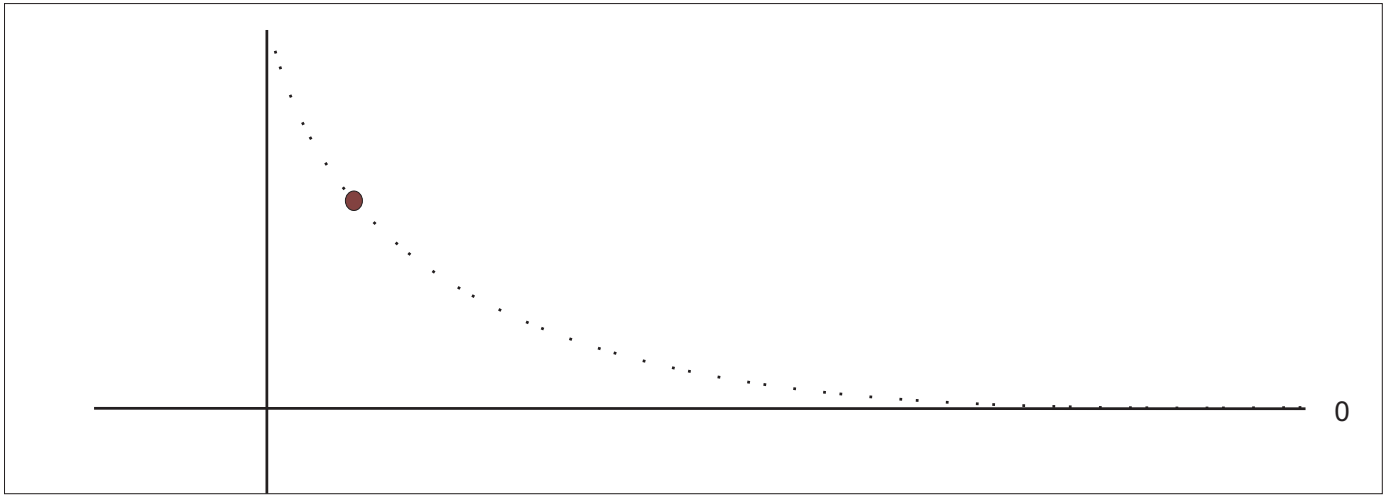

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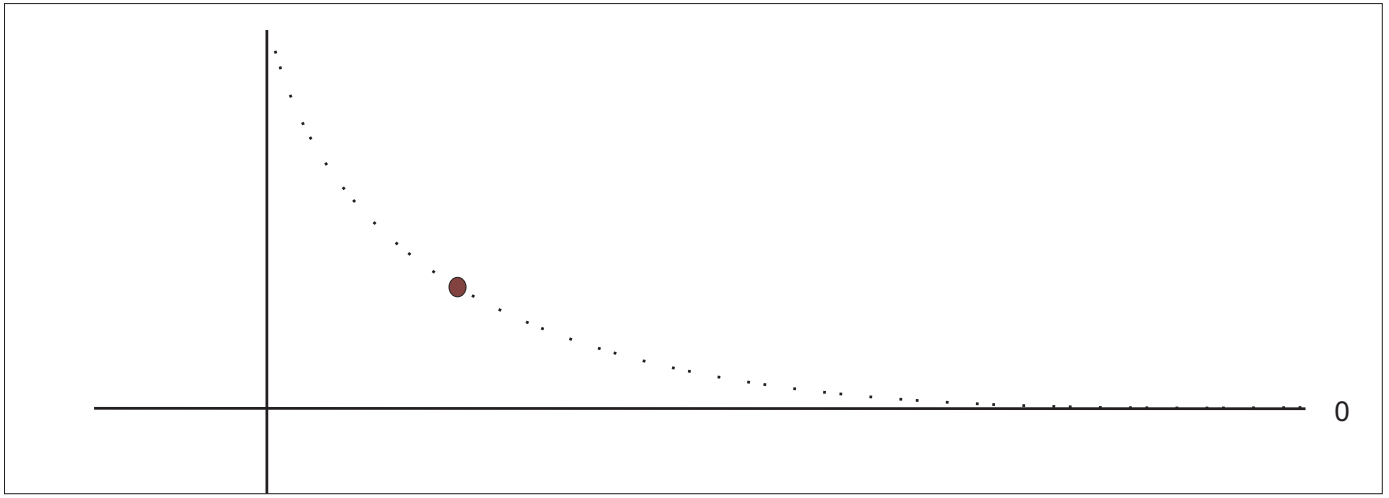

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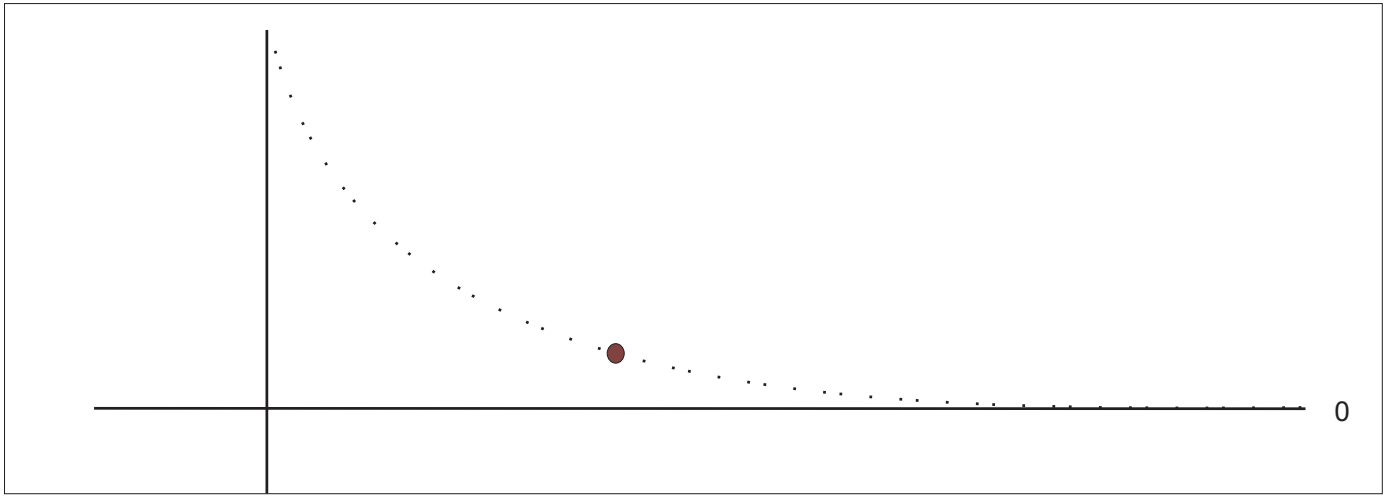

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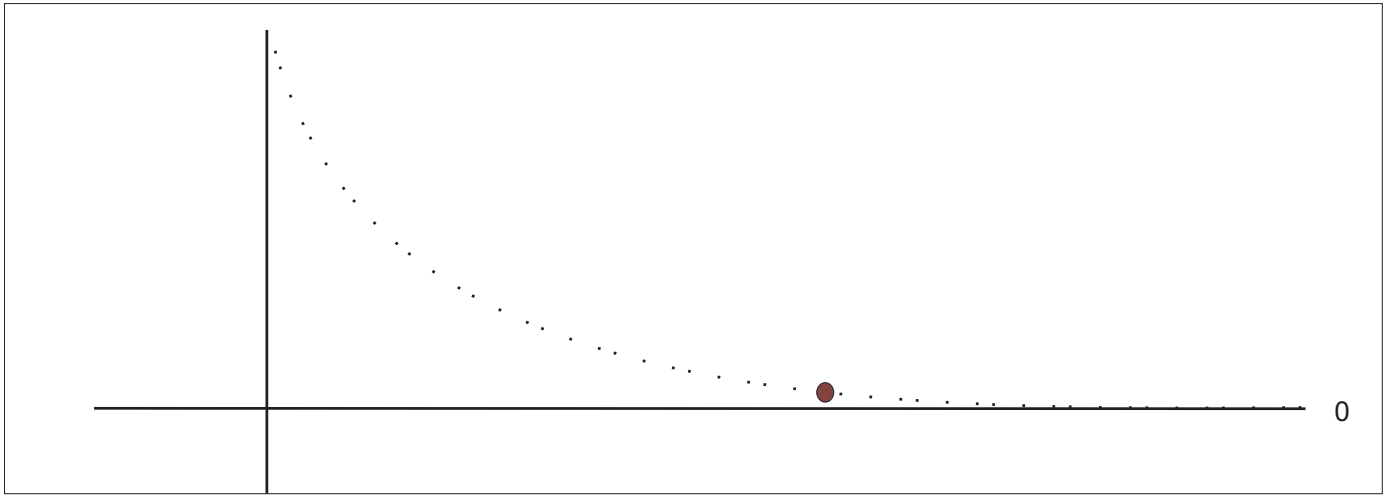

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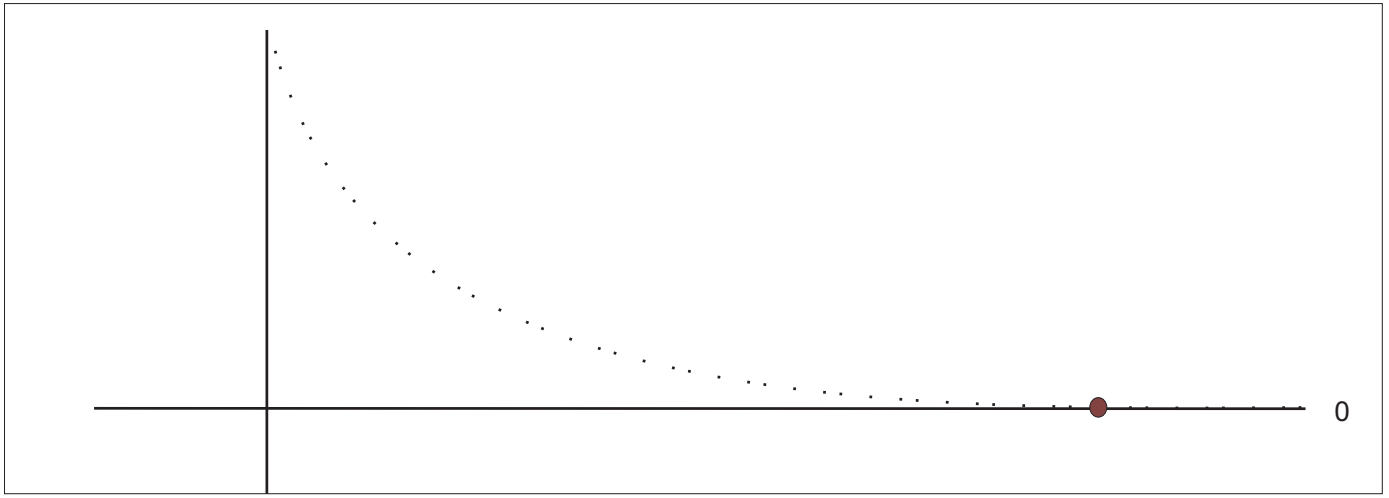

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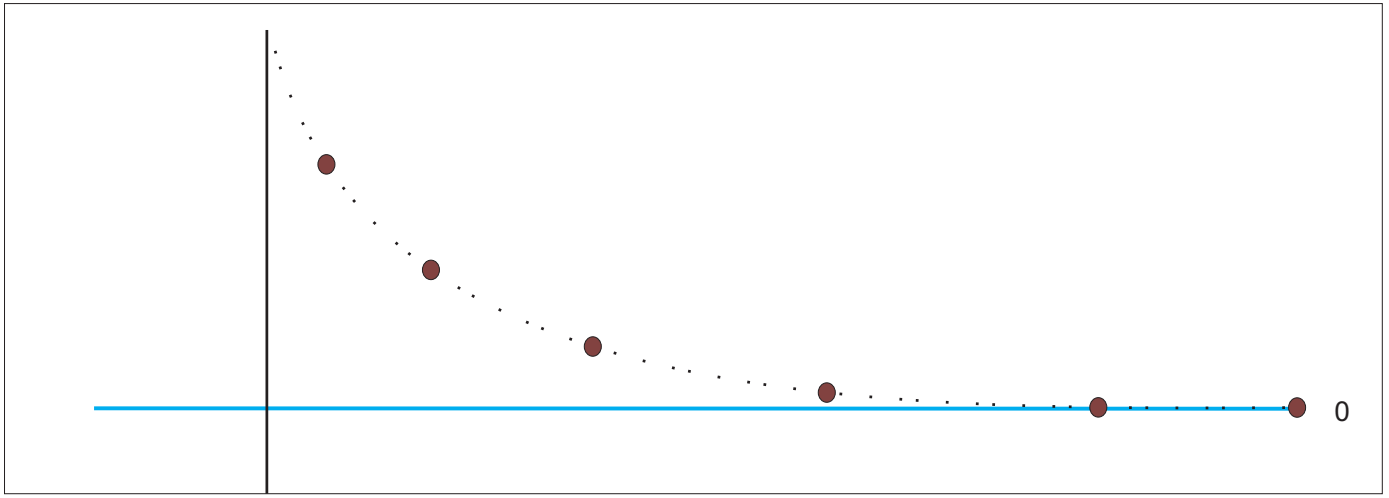

Again, consider $\{\frac{1}{n}\}$. As n *gets larger and larger* the terms get *closer and closer to* 0.

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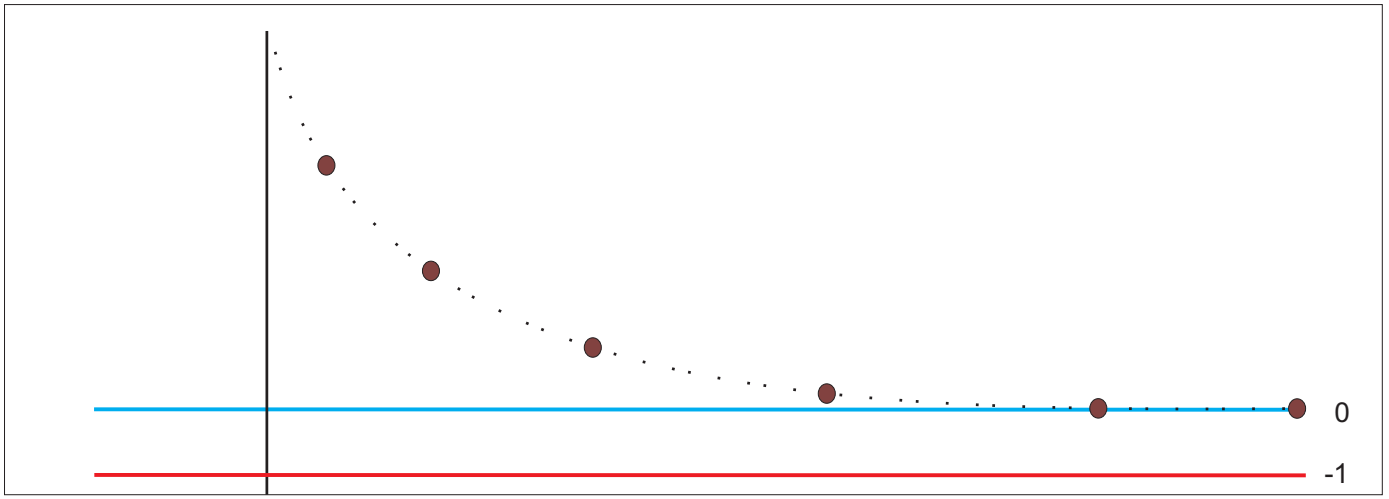

Again, consider $\{\frac{1}{n}\}$. As n *gets larger and larger* the terms get *closer and closer to* 0. **But they also get closer and closer to** -1**.**

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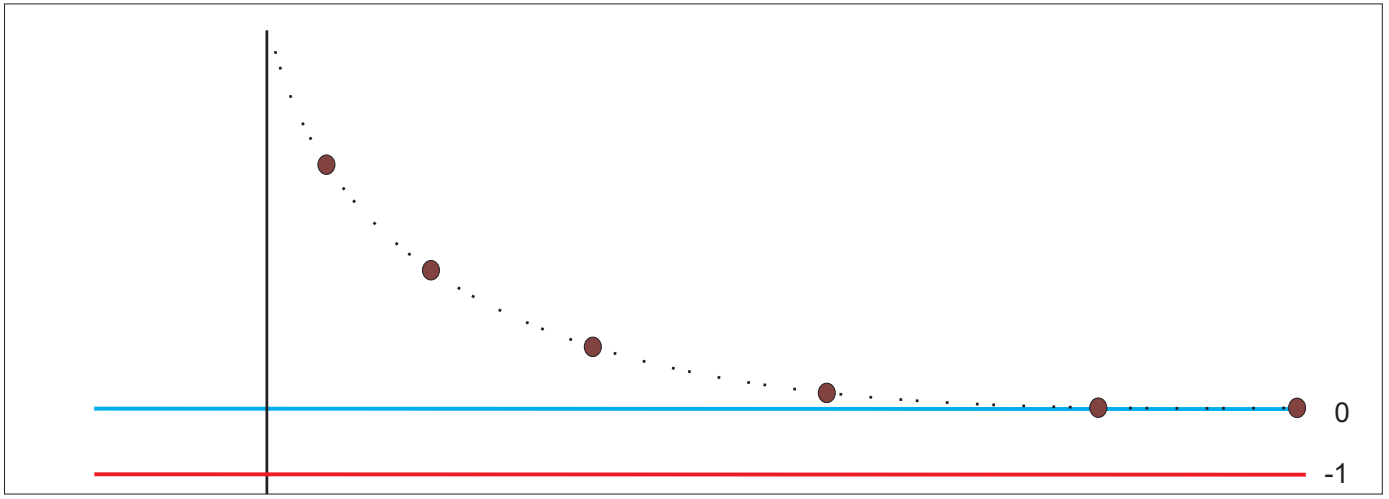

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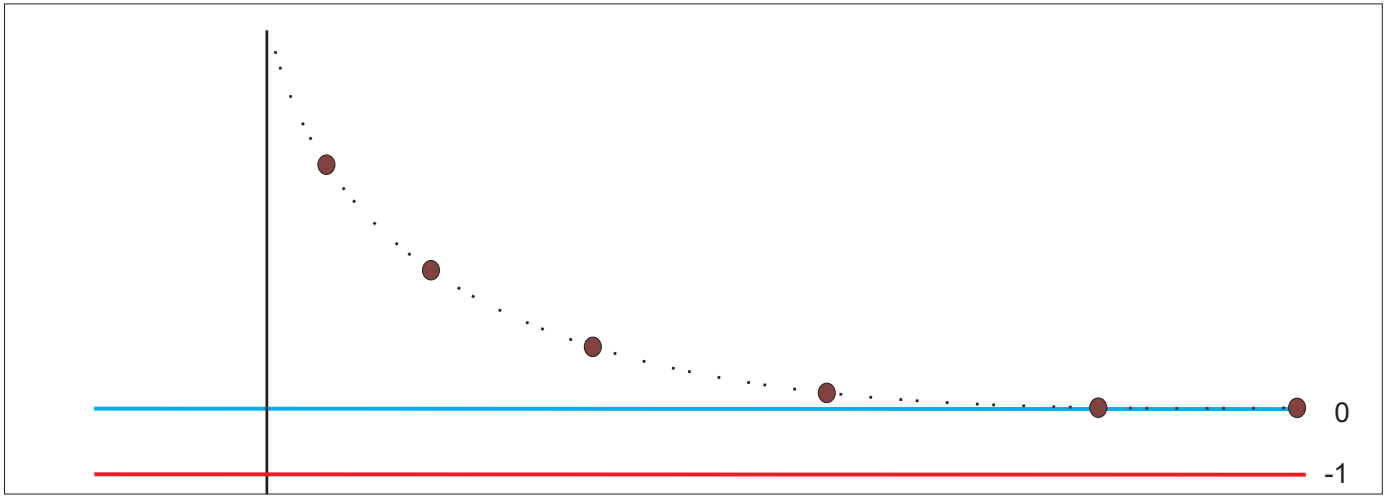

Question: What is it about 0 that makes us call it the limit of $\{\frac{1}{n}\}$ but we do not call -1 the limit as well?

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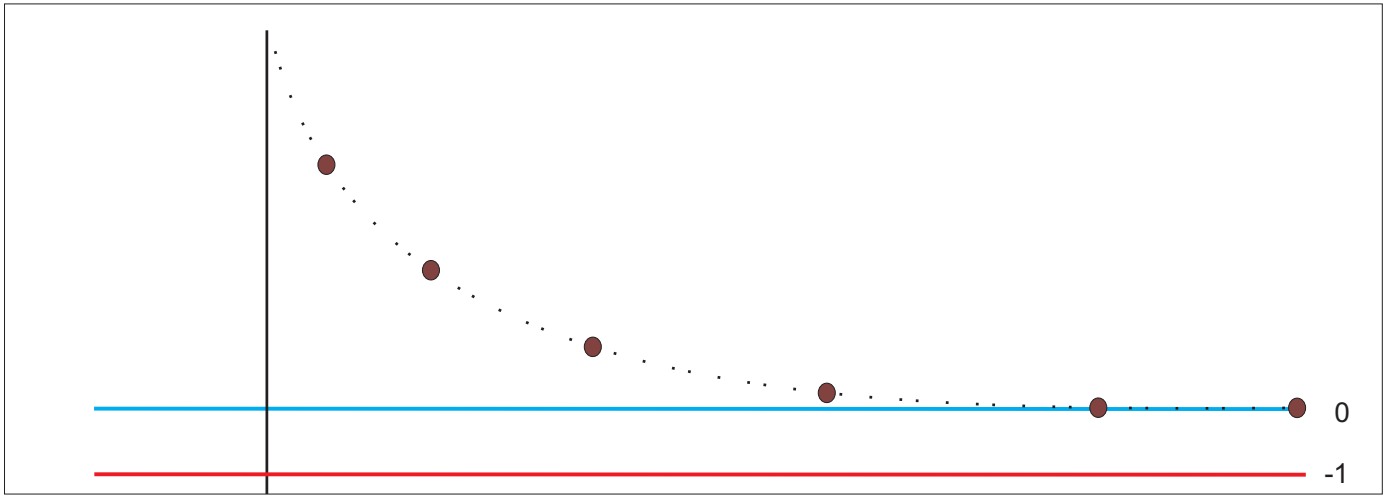

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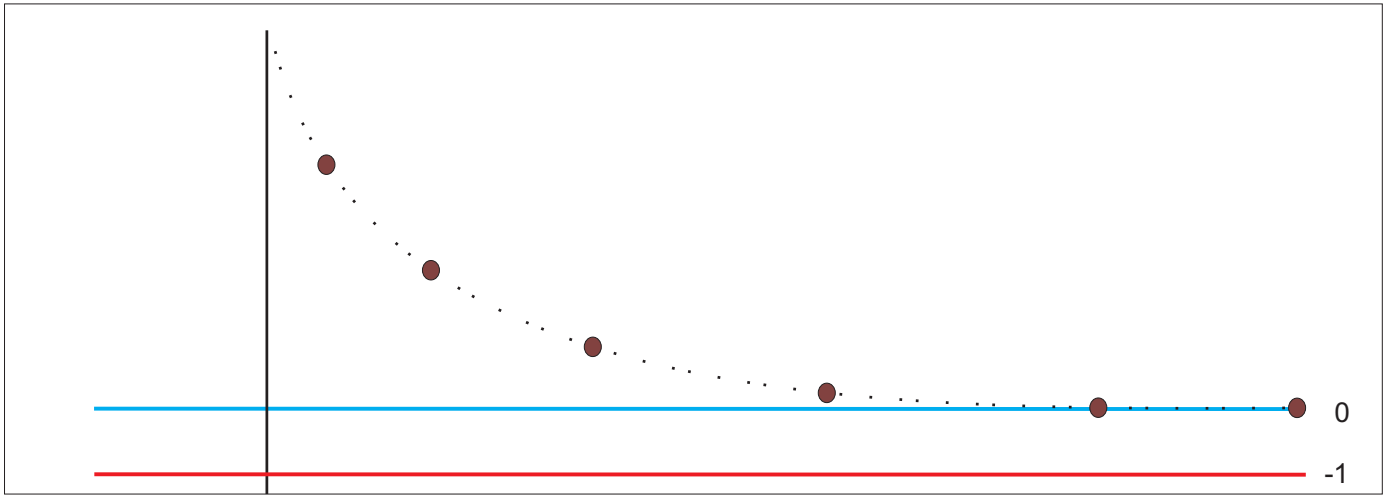

Question: What is it about 0 that makes us call it the limit of $\{\frac{1}{n}\}$ but we do not call -1 the limit as well?

Answer: The terms of $\{\frac{1}{n}\}$ approximate 0 as **closely as we would like** when n is large enough,

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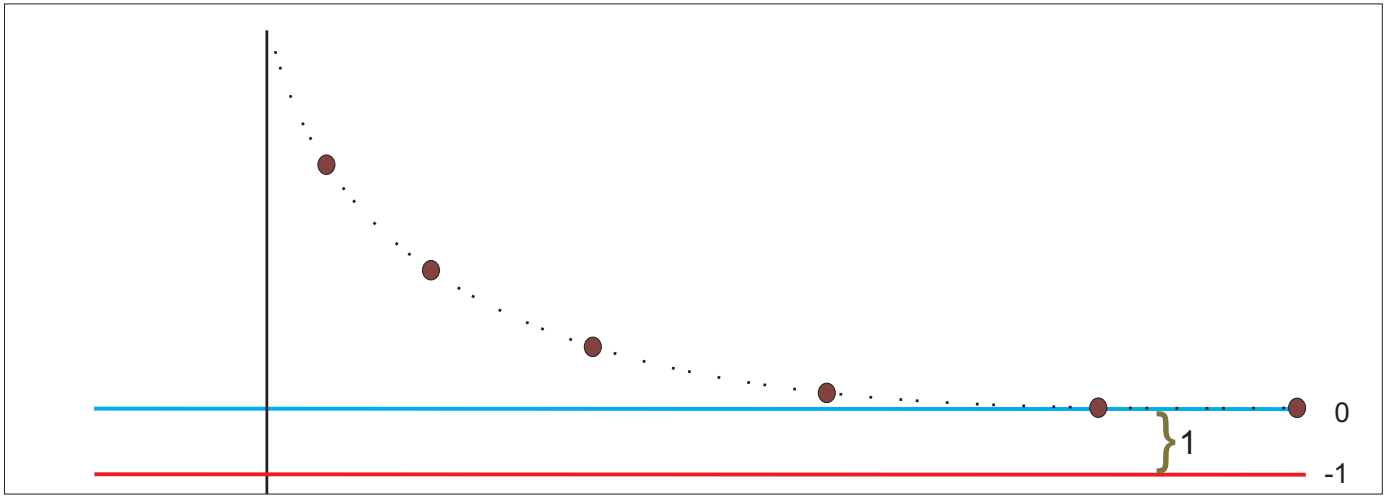

Question: What is it about 0 that makes us call it the limit of $\{\frac{1}{n}\}$ but we do not call -1 the limit as well?

Answer: The terms of $\{\frac{1}{n}\}$ approximate 0 as **closely as we would like** when n is large enough, but they never even get within 1 unit of -1.

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Definition: (New Heuristic Definition)

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We say that L is the *limit* of the sequence $\{a_n\}$ as n goes to infinity,

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We say that L is the *limit* of the sequence $\{a_n\}$ as n goes to infinity, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\mid a_n - L \mid < \epsilon.$$

Definition of a limit

Definition: (New Heuristic Definition)

We say that L is the *limit* of the sequence $\{a_n\}$ as n goes to infinity,if no matter what positive tolerance $\epsilon > 0$ we are given, we can find a cutoff $N \in \mathbb{N}$, such that the terms a_n approximate L with **error** less than ϵ provided that $n \geq N$.

Definition: (Formal Definition)

We say that L is the *limit* of the sequence $\{a_n\}$ as n goes to infinity, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\mid a_n - L \mid < \epsilon.$$

In this case we write

$$\lim_{n \to \infty} a_n = L.$$

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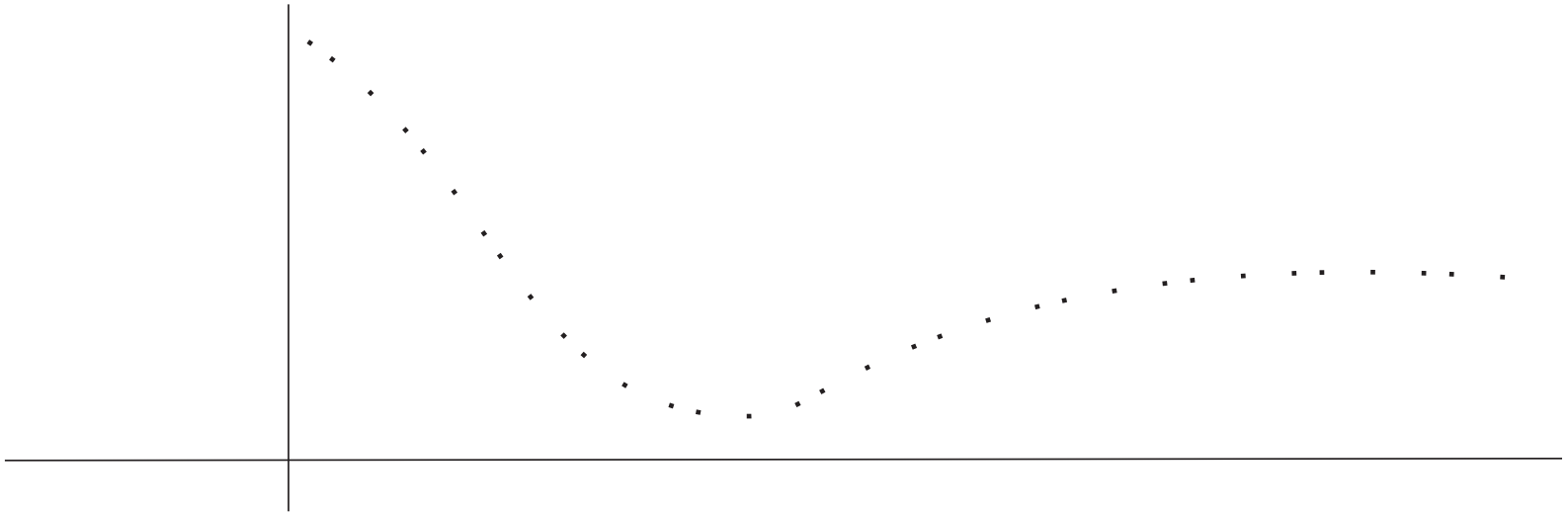

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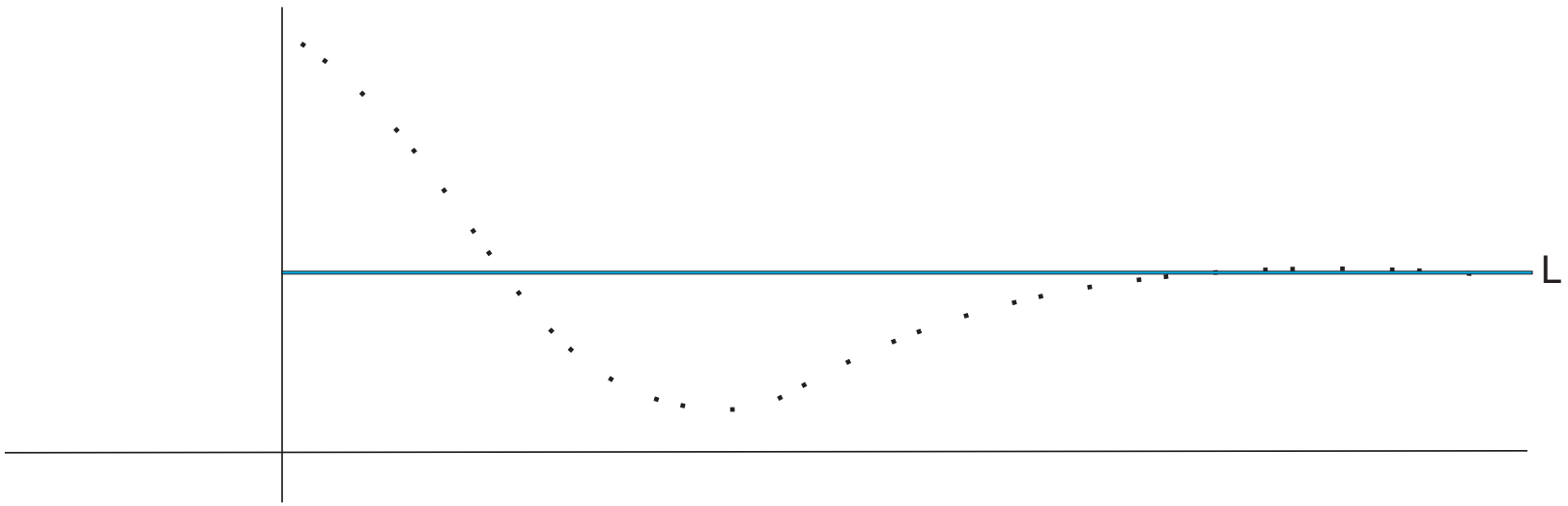

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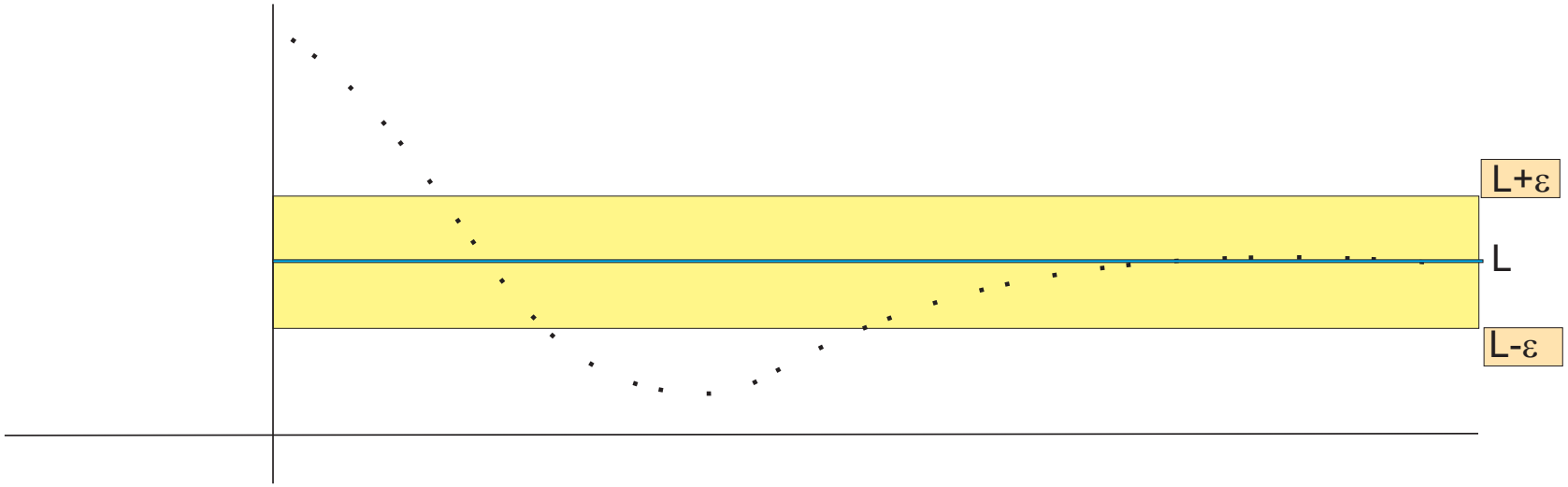

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2. Specify the error $\epsilon > 0$.

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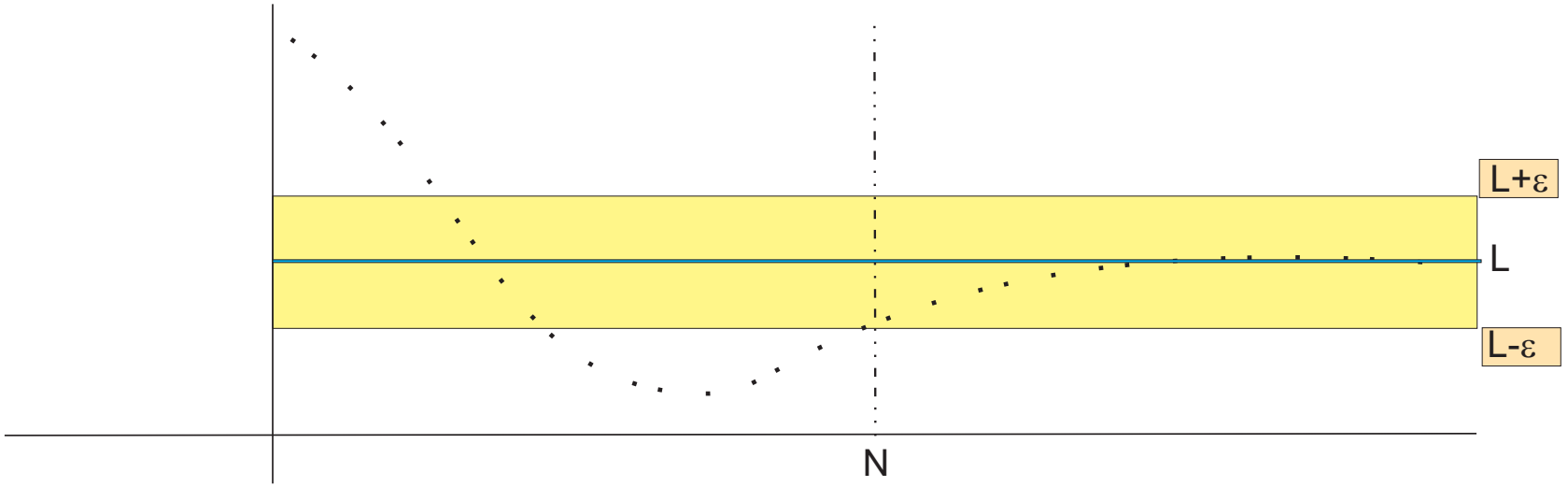

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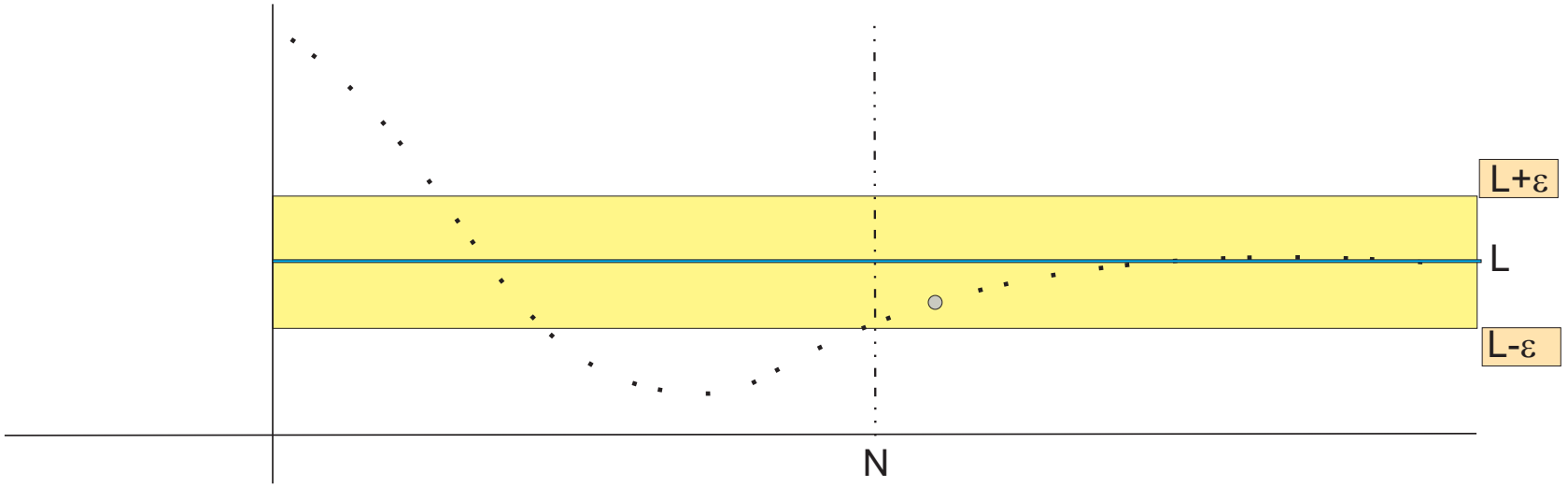

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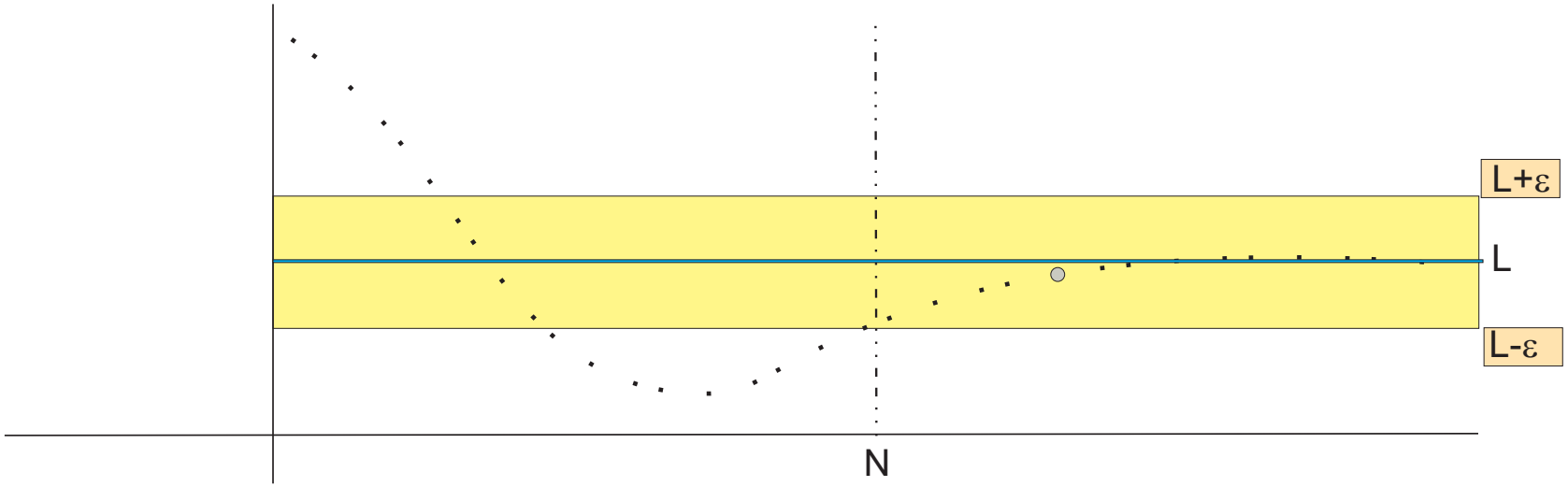

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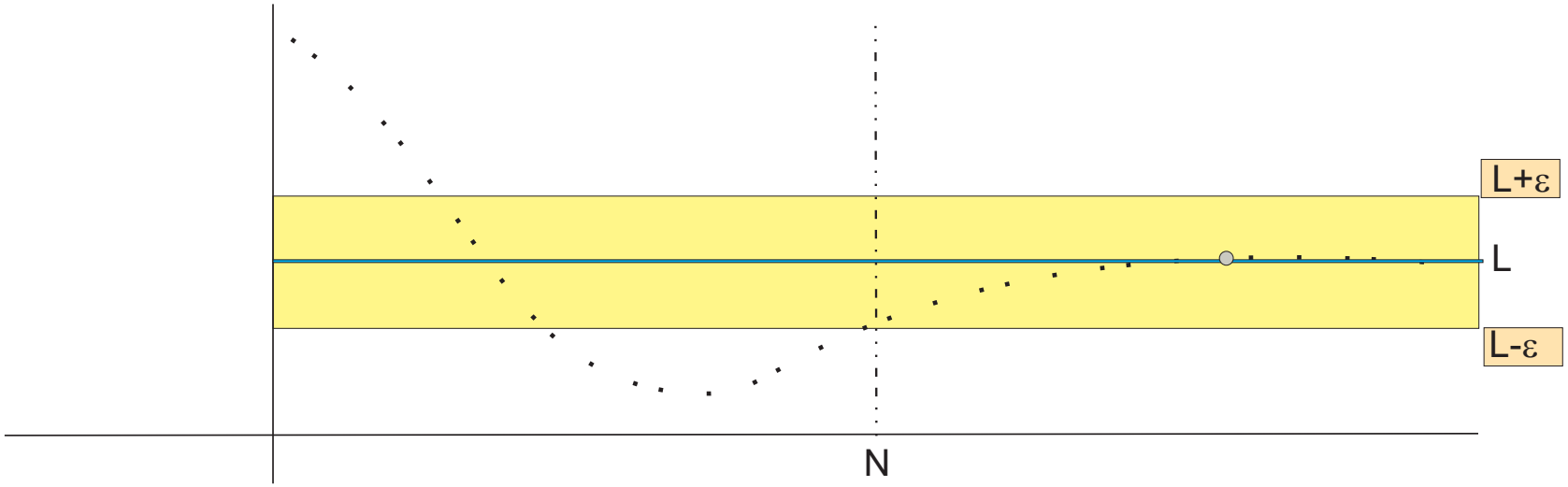

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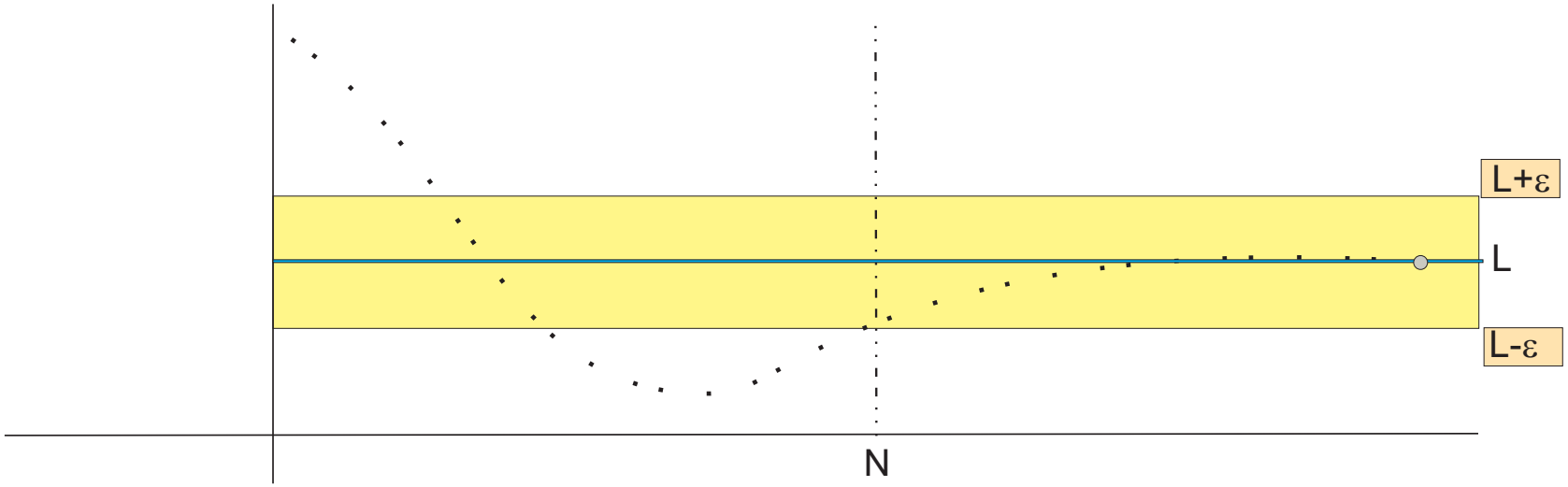

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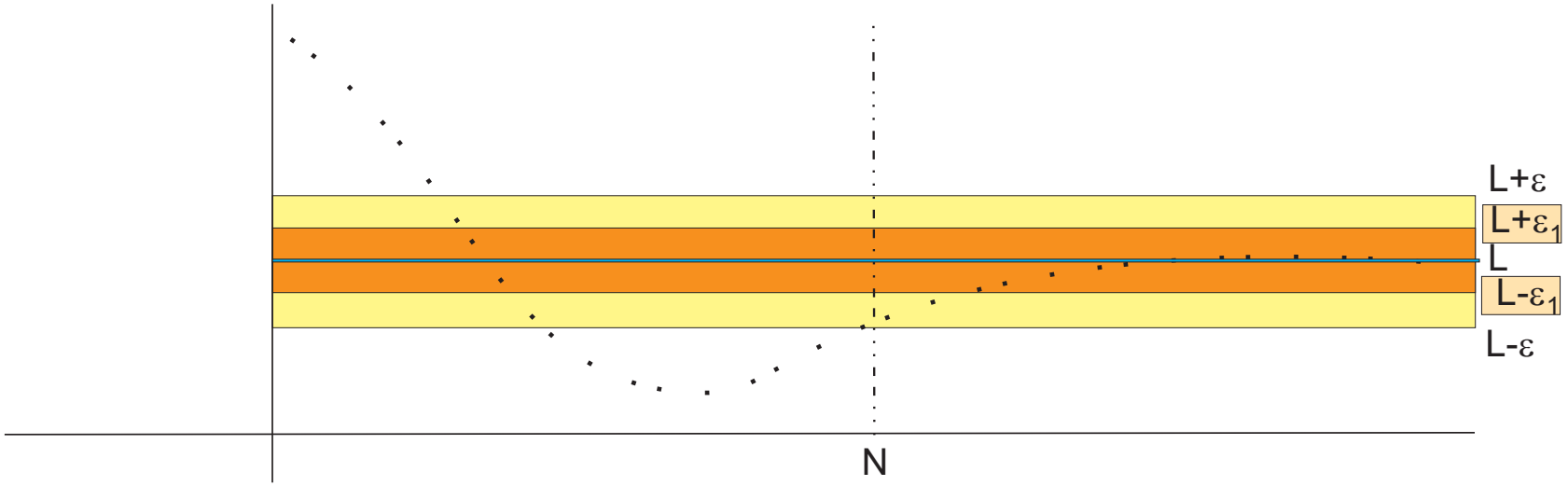

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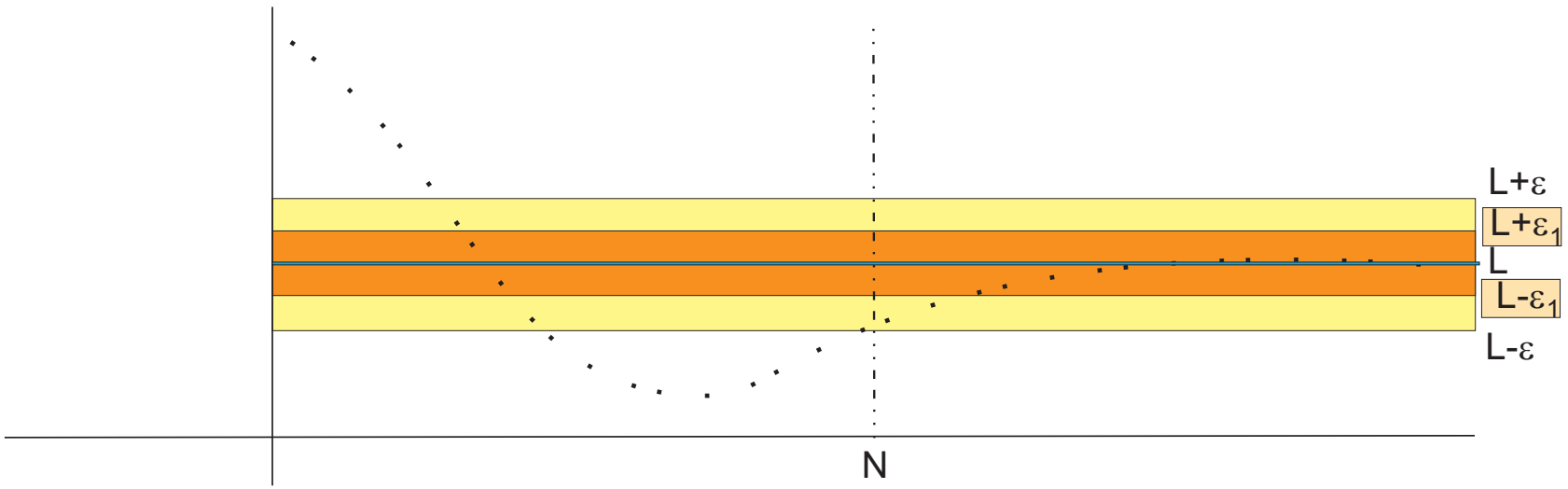

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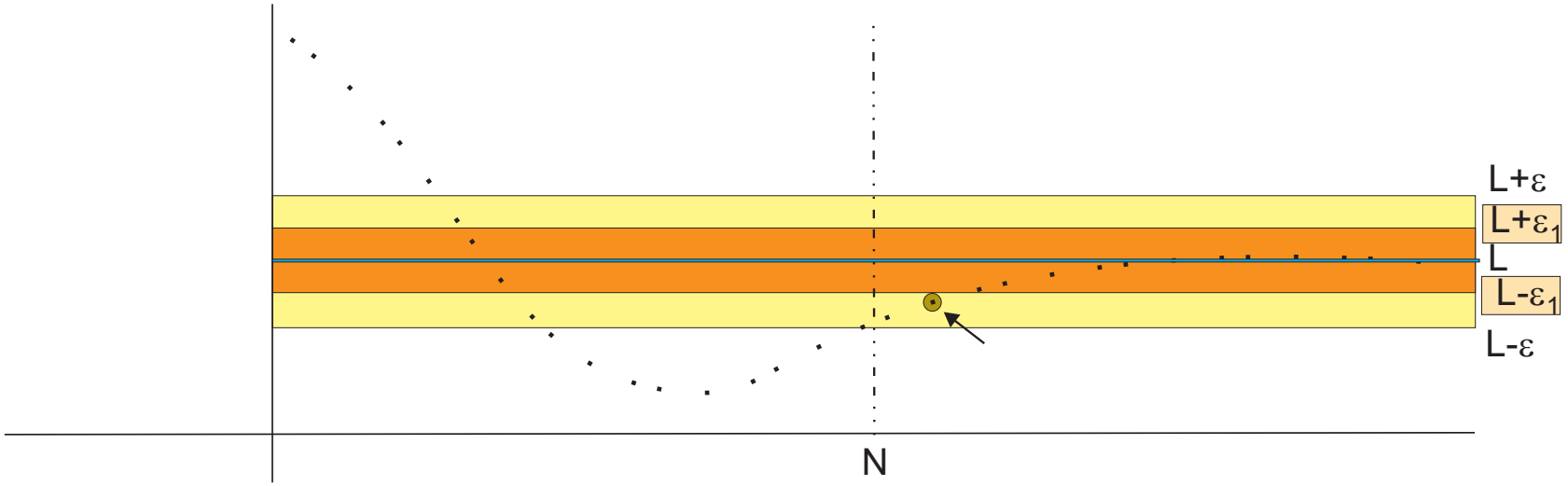

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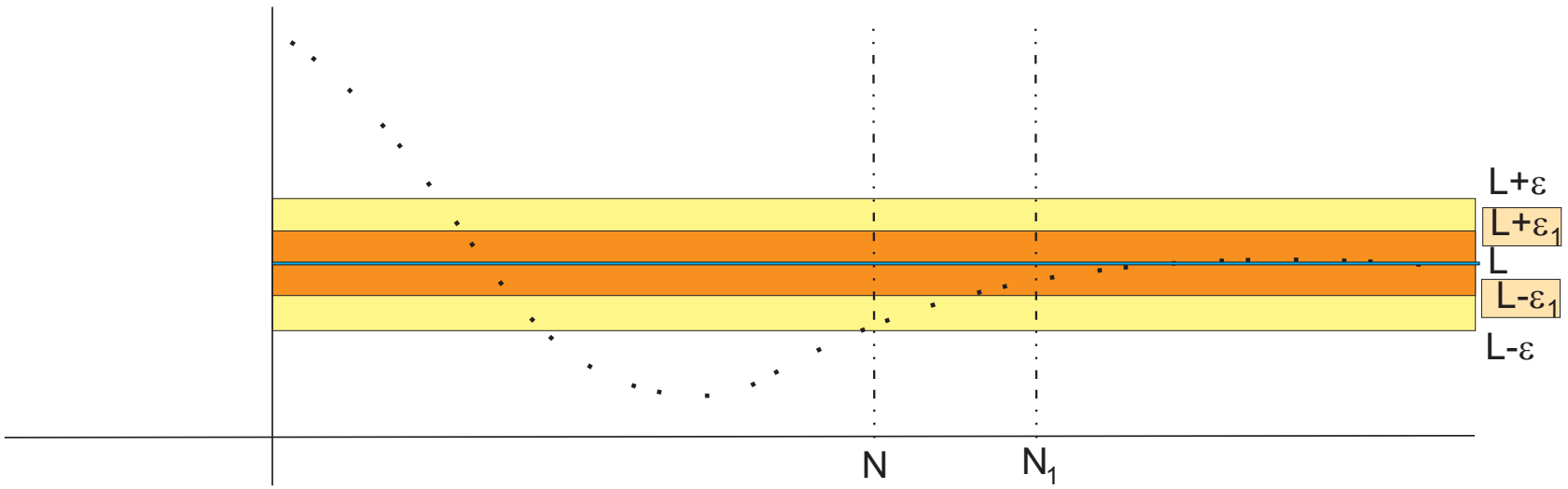

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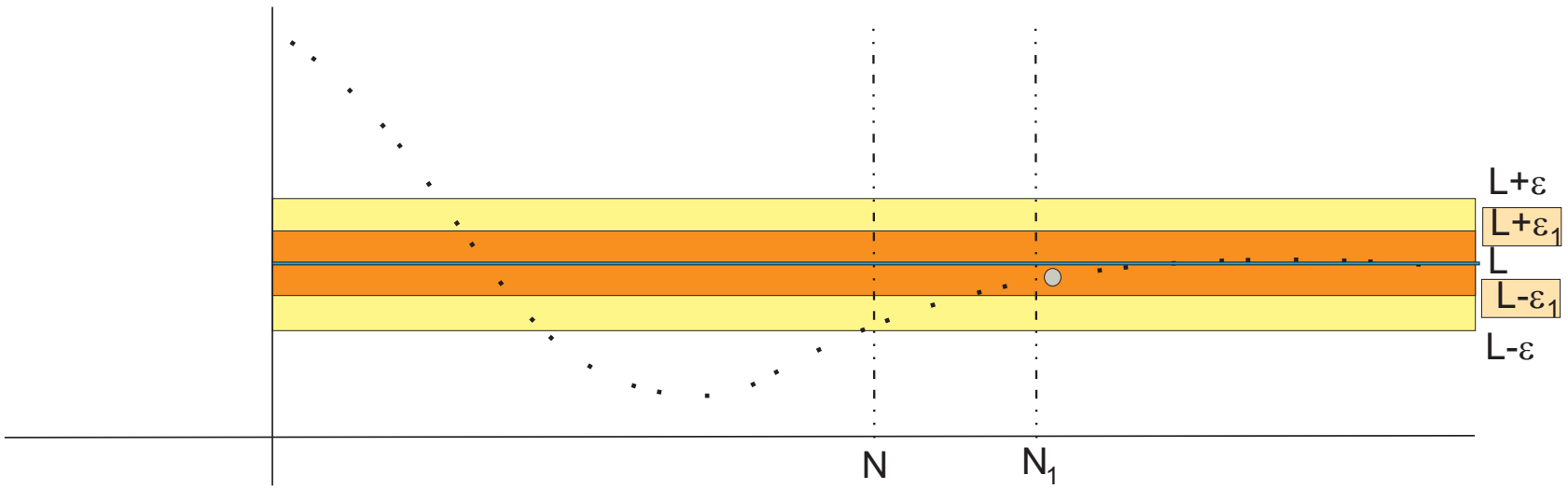

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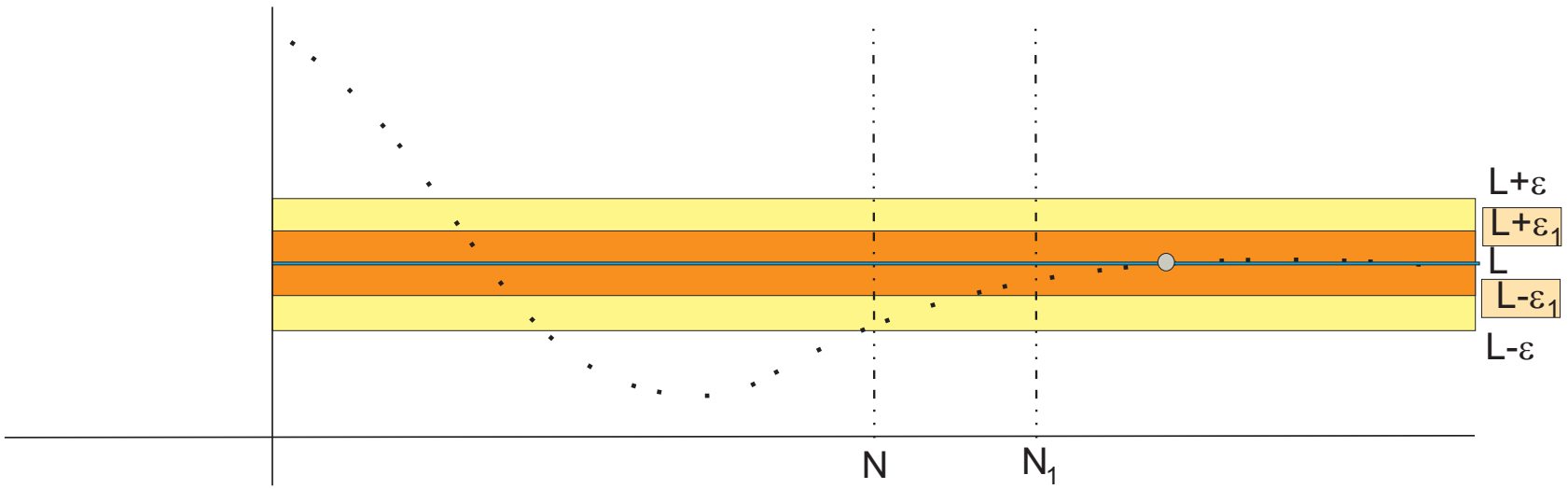

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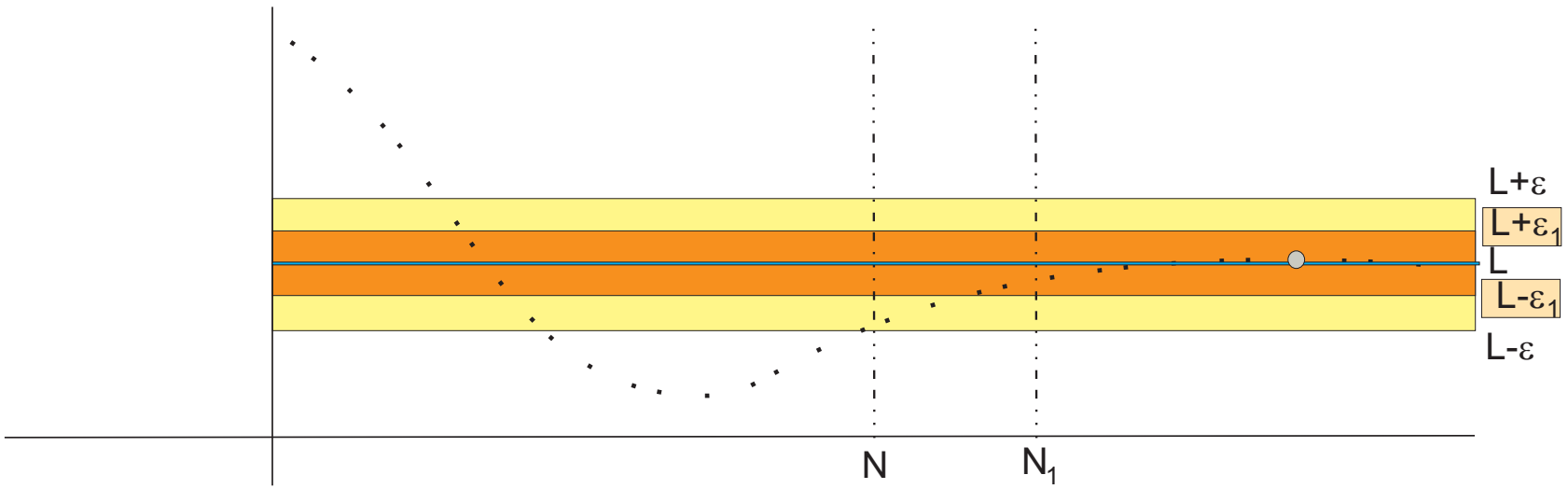

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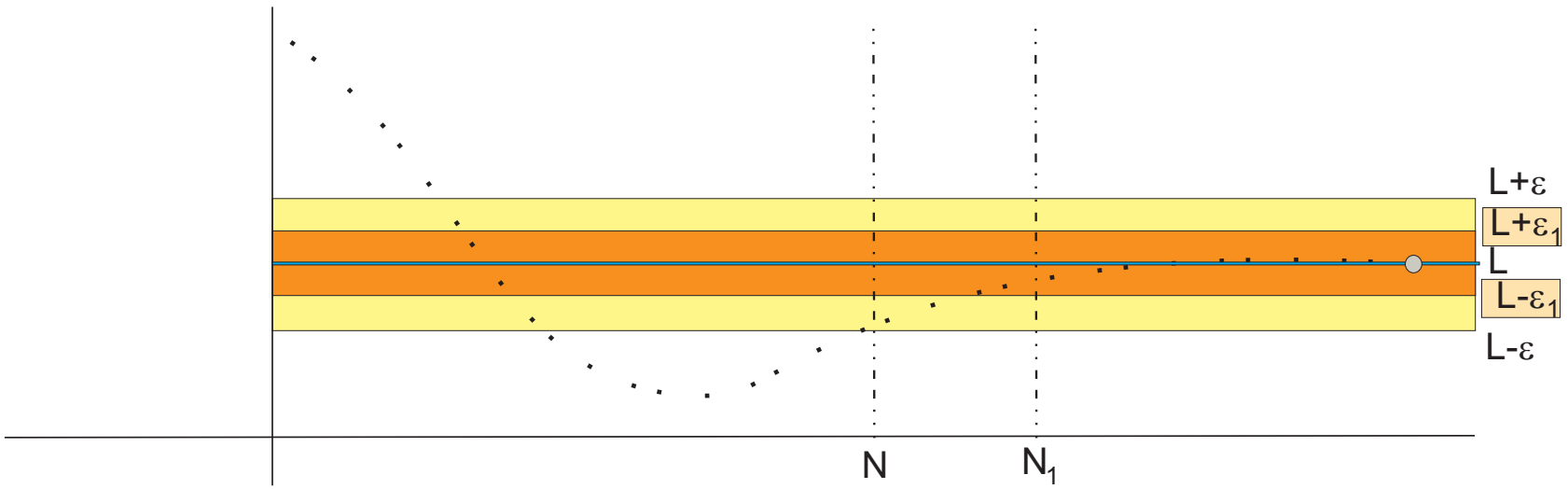

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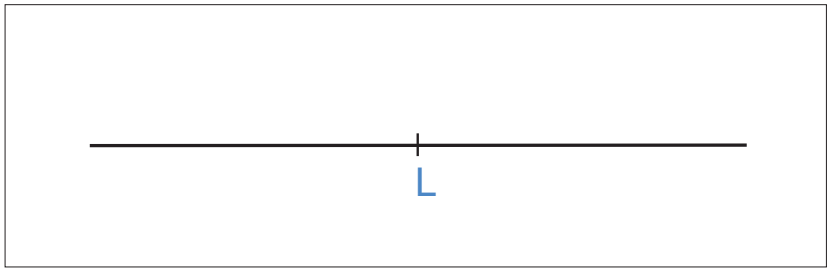

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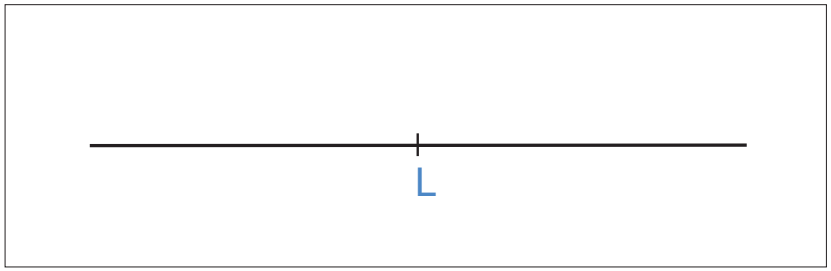

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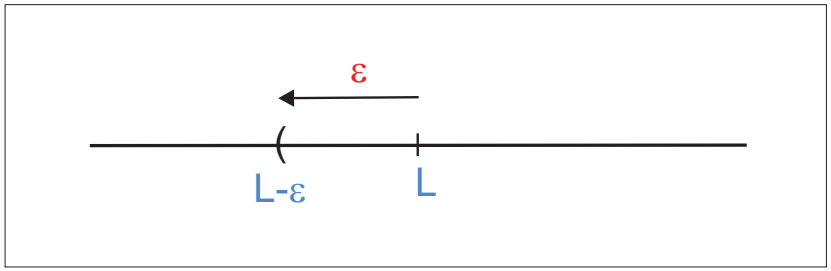

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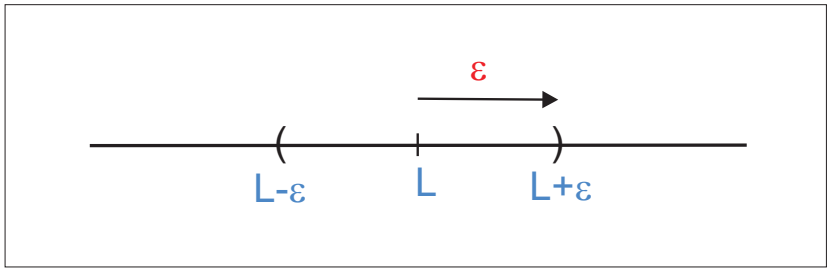

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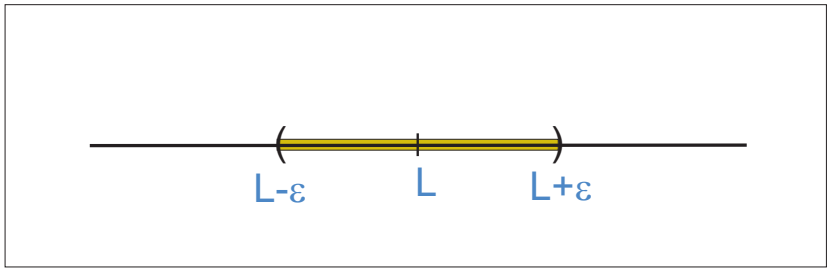

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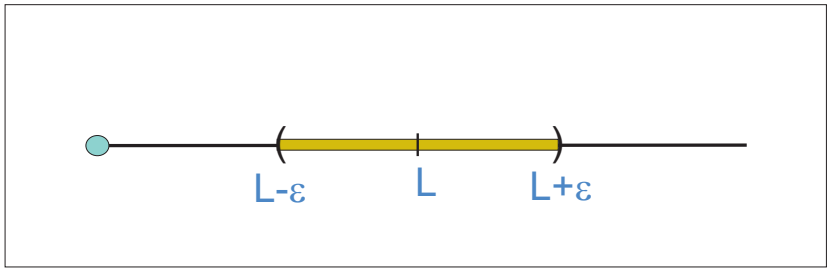

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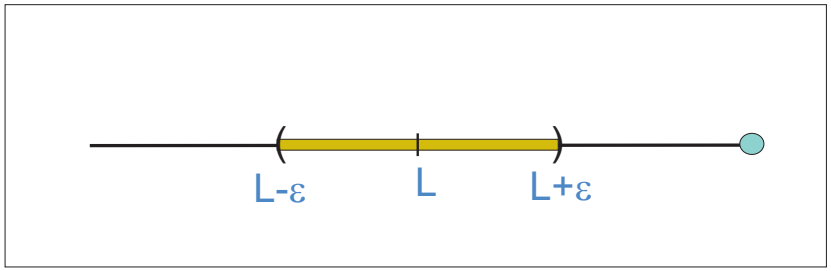

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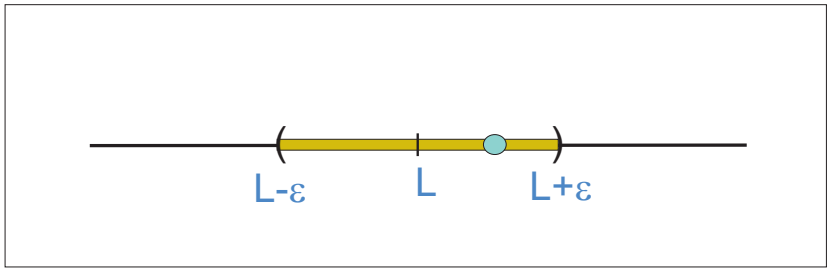

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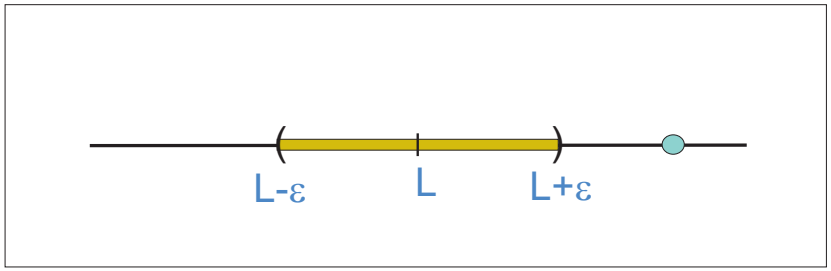

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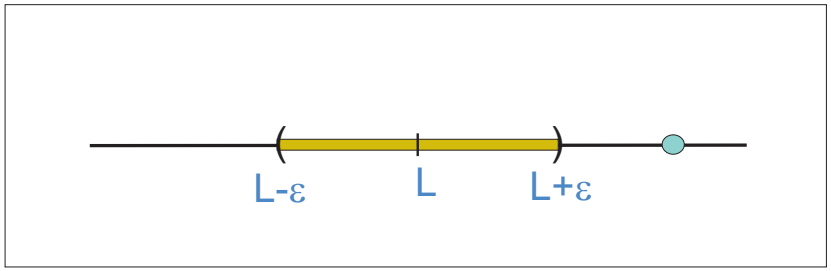

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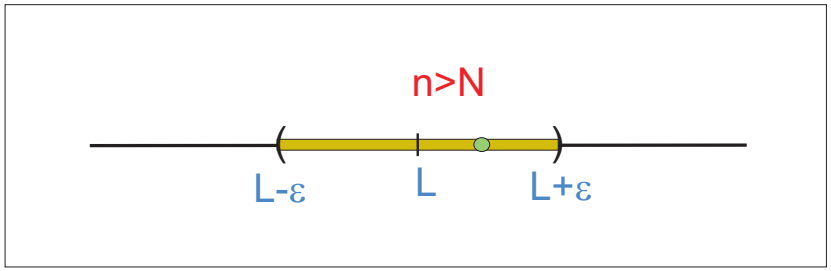

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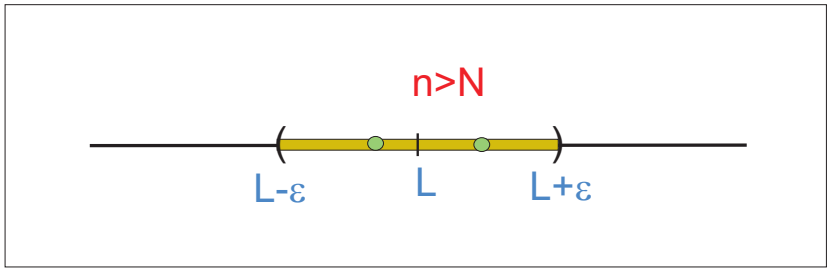

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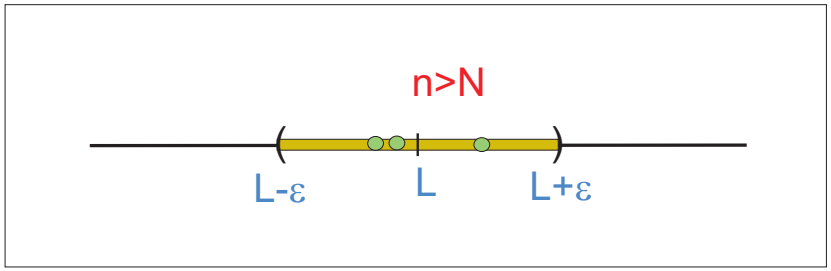

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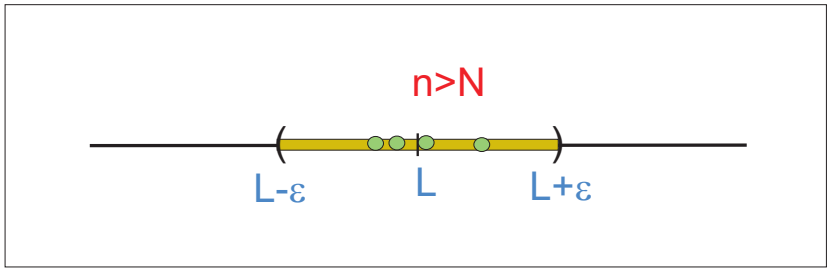

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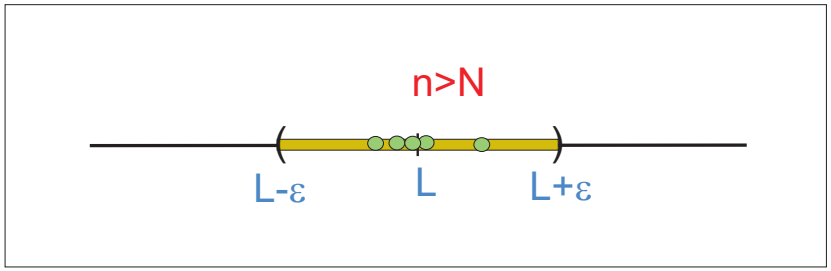

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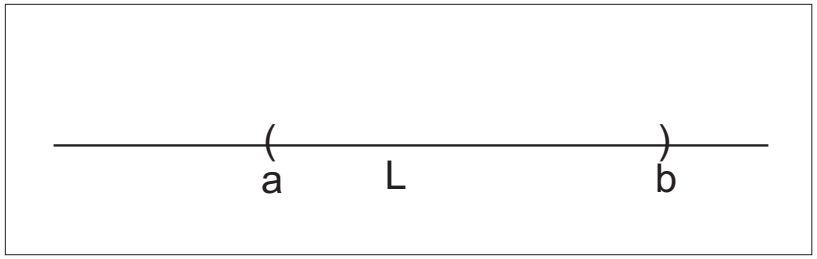

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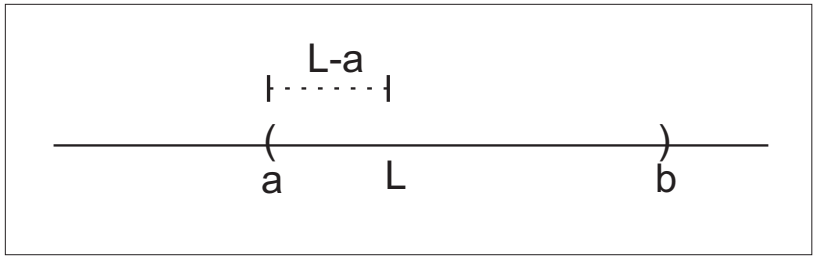

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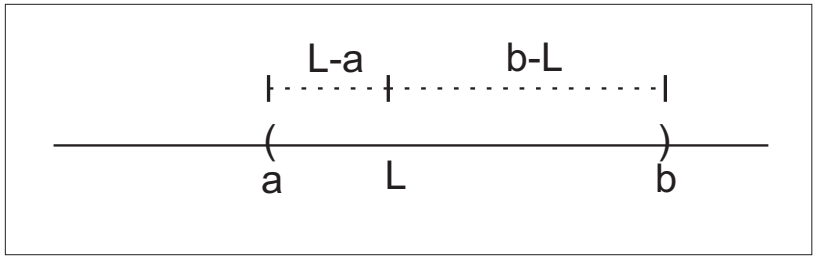

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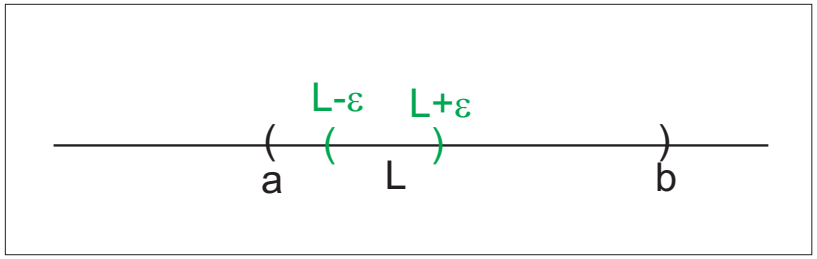

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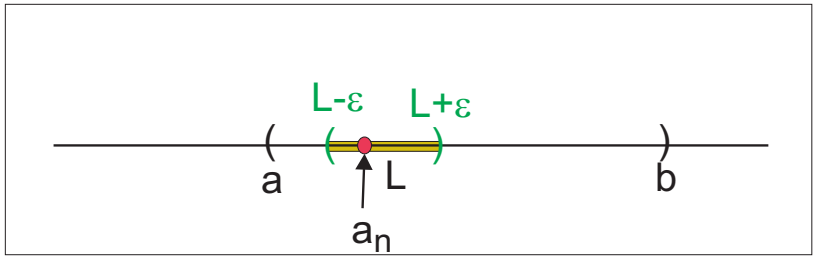

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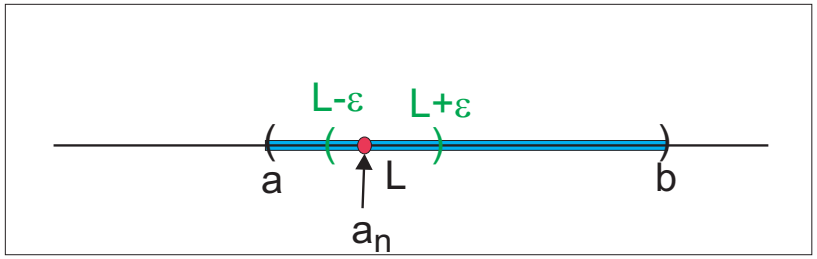

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If n is large enough, then $a_n \in (L - \epsilon, L + \epsilon)$ and hence

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1. $\lim_{n\to\infty} a_n = L$.
2. Every interval $(L-\epsilon, L+\epsilon)$ contains a **tail** of $\{a_n\}$.

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6. Changing finitely many terms in $\{a_n\}$ does not affect convergence.