Definition of a limit

Recall:

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We say that $L$ is the limit of the sequence $\{a_n\}$ as $n$ goes to infinity,
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We say that $L$ is the limit of the sequence $\{a_n\}$ as $n$ goes to infinity, if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n > N$, then $|a_n - L| < \epsilon$. In this case we write $\lim_{n \to \infty} a_n = L$. We may also say $\{a_n\}$ converges to $L$ and write $a_n \to L$. If no such $L$ exists, we say that $\{a_n\}$ diverges.
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**Definition:** (Formal Definition of a Limit:)
We say that \( L \) is the limit of the sequence \( \{a_n\} \) as \( n \) goes to infinity, if for every \( \epsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that if \( n > N \), then

\[
| a_n - L | < \epsilon.
\]

In this case we write

\[
\lim_{n \to \infty} a_n = L.
\]

We may also say \( \{a_n\} \) converges to \( L \) and write \( a_n \to L \).

If no such \( L \) exists, we say that \( \{a_n\} \) diverges.
Example: Show that $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$. 
**Example:** Show that \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). Let \( \epsilon > 0 \).
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\[ \epsilon \quad \frac{1}{\epsilon^2} \quad x \quad \frac{1}{\epsilon^2} \]

\[ x > \frac{1}{\epsilon^2} \]
Example: Show that \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). Let \( \epsilon > 0 \).

\[
\epsilon^2 > \frac{1}{x}
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Example: Show that \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). Let \( \epsilon > 0 \).

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x > \frac{1}{\epsilon^2} \Rightarrow \epsilon^2 > \frac{1}{x} \Rightarrow \epsilon > \frac{1}{\sqrt{x}}.
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Example: Show that \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). Let \( \epsilon > 0 \).

- \( x > \frac{1}{\epsilon^2} \Rightarrow \epsilon^2 > \frac{1}{x} \Rightarrow \epsilon > \frac{1}{\sqrt{x}} \).
- If \( \frac{1}{\epsilon^2} < N \)
Example: Show that \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). Let \( \epsilon > 0 \).

- If \( \frac{1}{\epsilon^2} < N \leq n \)

\[
\begin{align*}
x > \frac{1}{\epsilon^2} & \implies \epsilon^2 > \frac{1}{x} \implies \epsilon > \frac{1}{\sqrt{x}}.
\end{align*}
\]
Example: Show that \( \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0 \). Let \( \epsilon > 0 \).

\[ x > \frac{1}{\epsilon^2} \Rightarrow \epsilon^2 > \frac{1}{x} \Rightarrow \epsilon > \frac{1}{\sqrt{x}}. \]

\[ \text{If } \frac{1}{\epsilon^2} < N \leq n \Rightarrow \left| \frac{1}{\sqrt{n}} - 0 \right| < \epsilon. \]
Example:
Consider \((-1)^{n+1}\) = \{1, -1, 1, -1, \ldots\}.
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\[
\lim_{n \to \infty} \{(-1)^{n+1}\} = 1?
\]
Example:

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Does \(\{(-1)^{n+1}\}\) have a limit?

Is \(\lim_{n \to \infty} \{(-1)^{n+1}\} = 1\)? Is \(\lim_{n \to \infty} \{(-1)^{n+1}\} = -1\)?
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Or both?
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**Answer:** Assume \( \lim_{n \to \infty} \{ (-1)^{n+1} \} = L \) and \( \epsilon = 0.5 \).
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Or both?

**Answer:** Assume \( \lim_{n \to \infty} \{(-1)^{n+1}\} = L \) and \( \epsilon = 0.5 \). Choose the cutoff \( N \) such that if \( n > N \) then

\[
| (-1)^{n+1} - L | < 0.5.
\]
Example

Pick $k_0 \in \mathbb{N}$ such that $2k_0 + 1 \geq N$. 
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$$a_{2k_0 + 1} = 1$$
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Pick $k_0 \in \mathbb{N}$ such that $2k_0 + 1 \geq N$. Then

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$$\implies |1 - L| < 0.5$$
Example

Pick \( k_0 \in \mathbb{N} \) such that \( 2k_0 + 1 \geq N \). Then

\[
a_{2k_0+1} = 1 \\
\implies |1 - L| < 0.5 \\
\implies L \in (0.5, 1.5)
\]
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Pick \( k_1 \in \mathbb{N} \) such that \( 2k_1 \geq N \).
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$$\Rightarrow | -1 - L | < 0.5$$
Pick \( k_1 \in \mathbb{N} \) such that \( 2k_1 \geq N \). Then

\[
\begin{align*}
a_{2k_1} &= -1 \\
\Rightarrow \quad | -1 - L | &< 0.5 \\
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\end{align*}
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Pick $k_1 \in \mathbb{N}$ such that $2k_1 \geq N$. Then

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Hence $L \in (-1.5, -0.5)$ and $L \in (0.5, 1.5)$ which is impossible.
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Hence \( L \in (-1.5, -0.5) \) and \( L \in (0.5, 1.5) \) which is impossible.
Therefore, \( \{(-1)^{n+1}\} \) has no limit!
Uniqueness of Limits

**Problem:** Can \( \{a_n\} \) have two different limits?
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Assume \( \lim_{n \to \infty} a_n = L \)
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Problem: Can \{a_n\} have two different limits? 
Assume \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} a_n = M \) with \( L < M \).
Problem: Can \( \{a_n\} \) have two different limits?

Assume \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} a_n = M \) with \( L < M \). Consider \( \frac{M+L}{2} \).
**Problem:** Can \( \{a_n\} \) have two different limits?

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Let \( \epsilon = \frac{M-L}{2} \).
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Consider \( a_{n_0} \).
**Problem:** Can \( \{a_n\} \) have two different limits?

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Let \( \epsilon = \frac{M-L}{2} \).

Consider \( a_{n_0} \). If \( n_0 \) is large enough, then

\[
a_{n_0} \in (M - \epsilon, M + \epsilon)
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**Problem:** Can \( \{a_n\} \) have two different limits?

Assume \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} a_n = M \) with \( L < M \). Consider \( \frac{M+L}{2} \).

Let \( \epsilon = \frac{M-L}{2} \).

Consider \( a_{n_0} \). If \( n_0 \) is large enough, then

\[
 a_{n_0} \in (M - \epsilon, M + \epsilon)
\]

and

\[
 a_{n_0} \in (L - \epsilon, L + \epsilon)
\]
Uniqueness of Limits

Problem: Can \( \{a_n\} \) have two different limits?

Assume \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} a_n = M \) with \( L < M \). Consider \( \frac{M+L}{2} \).

Let \( \epsilon = \frac{M-L}{2} \).

Consider \( a_{n_0} \). If \( n_0 \) is large enough, then

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and

\[
a_{n_0} \in (L - \epsilon, L + \epsilon)
\]

which is impossible!
Uniqueness of Limits

**Theorem:** (Uniqueness of Limits)
Assume that \( \lim_{n \to \infty} a_n = L \) and \( \lim_{n \to \infty} a_n = M \). Then

\[
L = M.
\]
Often, it is difficult to tell if a sequence converges or if so, what its limit might be.
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**Example:** Consider the recursively defined sequence

\[ a_1 = 1 \quad a_{n+1} = \cos(a_n). \]
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Does \( \{a_n\} \) converge?
Often, it is difficult to tell if a sequence converges or if so, what its limit might be.

**Example:** Consider the recursively defined sequence

\[ a_1 = 1 \quad a_{n+1} = \cos(a_n). \]

Does \( \{a_n\} \) converge? If so, what is \( \lim_{n \to \infty} a_n \)?
Example

$$a_1 = 1,$$
Example

\[ a_1 = 1, \]

\[ \cos(x) \quad y = x \]
Example

\[ a_1 = 1, \]

Graph showing \( y = \cos(x) \) and \( y = x \) intersecting at \( a_1 \)
Example

\[ a_1 = 1, \]

\[ 0.7314040424, 0.7442373549, 0.7356047404, 0.7414250866, \\
0.7375068905, 0.7401473356, 0.7383692041, 0.7395672022, \\
0.7387603199, 0.7393038924, 0.7389377567, 0.7391843998, \ldots \]
Example

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
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\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
\[ a_3 = 0.8575532158, \]
\[ a_4 = 0.6542897905, \]
Example

\[
\begin{align*}
a_1 &= 1, \\
a_2 &= 0.5403023059, \\
a_3 &= 0.8575532158, \\
a_4 &= 0.6542897905, \\
a_5 &= 0.7934803587, \\
\end{align*}
\]
Example

\[ a_1 = 1, \]
\[ a_2 = 0.5403023059, \]
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\[ a_6 = 0.7013687737, \]

\( y = x \)

\( \cos(x) \)
Example

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  a_5 &= 0.7934803587, \\
  a_6 &= 0.7013687737, \\
  a_7 &= 0.7639596829, \\
  a_8 &= 0.7221024250, \\
  a_9 &= 0.7504177618,
\end{align*} \]
Example

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\[ \ldots \]
Example

\[ a_{72} = 0.7390851332, \ a_{73} = 0.7390851332, \ a_{74} = 0.7390851332, \]
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suggests that \( \{a_n\} \) converges to some \( L \).
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suggests that \( \{a_n\} \) converges to some \( L \).

In fact,

\[ \cos(L) = L. \]