UNIVERSITY OF WATERLOO
University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles
November 17, 2010
Sequences

Sequences
A list of numbers or objects in which all terms are related by a rule is called a **pattern** or **sequence**. Each number in the list is called a **term**. All terms in sequences may be expressed inside curly brackets.

Example

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, ...\}$$

We can describe this sequence in multiple ways:

1. Increasing even positive integers.
2. Each new term is two more than the previous term.

Notation
Each number in the sequence will be expressed by t_n, where n is the term number in the sequence.

Example

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20...\}$$
$$t_3 = 6, \text{ 'The } 3^{rd} \text{ term in the sequence is 6'}$$

Recursive: defines each term in the sequence based on the previous term in the sequence. When defining a sequence recursively the first term(t_1) must be defined.

Example

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20...\}$$

Recursive: $t_1 = 2$,

$t_n =$ the previous term+2

$t_n = t_{n-1} + 2$

Exercise 1
First describe the pattern of the following sequences in words. Once you have determined the pattern, define the first term(t_1) and the general sequence(t_n).

a) $\{1, 3, 5, 7, 9, 11, 13, ...\}$

b) $\{5, 10, 15, 20, 25, 30, 35, ...\}$

c) $\{70, 63, 56, 49, 42, 35, 28, ...\}$

d) $\{100, 99, 97, 94, 90, 85, 79...\}$

Often we want to know a specific number in a sequence. We can determine specific numbers in a sequence by first determining the pattern then applying the pattern to find the required number.

Example
Find the 11th and 12th terms in the following sequence.

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20...\}$$

Pattern: Each new term adds two to the previous term.

$t_{11} =$ previous term $+2$
$t_{11} = t_{10} + 2$
$t_{11} = 20 + 2$
$t_{11} = 22$

$t_{12} =$ previous term $+2$
$t_{12} = t_{11} + 2$
$t_{12} = 22 + 2$
$t_{12} = 24$

Exercise 2
Find the 8th and 9th terms in each of the following sequences using the method above.

a) $\{1, 3, 5, 7, 9, 11, 13, ...\}$

b) $\{5, 10, 15, 20, 25, 30, 35, ...\}$

c) $\{70, 63, 56, 49, 42, 35, 28, ...\}$

d) $\{100, 99, 97, 94, 90, 85, 79...\}$

Arithmetic and Geometric Sequences

Arithmetic Sequence: a sequence where the difference between two successive terms in a sequence is constant. The difference between each term is known as the **common difference**.

Example

$$\{1, 7, 13, 19, 25, 31, 37, ...\}$$
$$7 - 1 = \mathbf{6}$$
$$13 - 7 = \mathbf{6}$$
$$19 - 13 = \mathbf{6}$$
$$\vdots$$

If we look at the above sequence we are able to find a pattern between the terms of the sequence and the first term.

$$\{1, 7, 13, 19, 25, 31, 37, ...\}$$
$$\{1, 1+6, 1+12, 1+18, 1+24, 1+30, 1+36, ...\}$$

Now lets write the second term in the equation as a product of 6.

$$\{1, 1+(6)(1), 1+(6)(2), 1+(6)(3), 1+(6)(4), 1+(6)(5), 1+(6)(6), ...\}$$

Are we able to generalize the number being multiplied by 6?

$$\begin{aligned} 1+(6)(1) &= 2^{nd} \text{ term} \\ 1+(6)(2) &= 3^{rd} \text{ term} \\ 1+(6)(3) &= 4^{th} \text{ term} \\ 1+(6)(4) &= 5^{th} \text{ term} \\ 1+(6)(5) &= 6^{th} \text{ term} \\ 1+(6)(6) &= 7^{th} \text{ term} \end{aligned}$$

The number being multiplied by 6 is one less than the term number. Therefore each term multiplies the common difference by one less than its term number, and adds the first term.

Geometric Sequence: a sequence where each term is found by multiplying the previous term by a non-zero constant. The constant being multiplied by is known as the **common ratio**.

Example

$$\{3, 6, 12, 24, 48, 96, 192, ...\}$$
$$3 \times \mathbf{2} = 6$$
$$6 \times \mathbf{2} = 12$$
$$12 \times \mathbf{2} = 24$$
$$\vdots$$

Exercise 3
Are the following arithmetic sequences, geometric sequences, or neither? If it is an arithmetic sequence find the common difference. If it is a geometric sequence find the common ratio.

a) $\{13, 24, 35, 46, 57, 68, 79, ...\}$

b) $\{7, 21, 63, 189, 567, 1701, 5103, ...\}$

c) $\{3, 16, 29, 34, 48, 56, ...\}$

d) $\{185, 173, 161, 149, 137, 125, 113, ...\}$

e) $\{512, 256, 128, 64, 32, 16, 8, ...\}$

f) $\{100, 89, 94, 73, 85, 61, 70, ...\}$

Special Sequences

Square Numbers

A **square number** is a number that results from multiplying a number by itself.

ie. $4 = 2 \times 2$, $9 = 3 \times 3$

Lets look at the sequence of square numbers and write each term as a product of squares.

$$\{1, 4, 9, 16, 25, 36, 49...\}$$
$$\{1 \times 1, 2 \times 2, 3 \times 3, 4 \times 4, 5 \times 5, 6 \times 6, 7 \times 7...\}$$

Are we able to find a pattern between the term number and the number being multiplied by itself?

$$1 \times 1 = 1^{st} \text{ term}$$
$$2 \times 2 = 2^{nd} \text{ term}$$
$$3 \times 3 = 3^{rd} \text{ term}$$
$$4 \times 4 = 4^{th} \text{ term}$$
$$5 \times 5 = 5^{th} \text{ term}$$
$$6 \times 6 = 6^{th} \text{ term}$$
$$7 \times 7 = 7^{th} \text{ term}$$

From above, we are able to see the number being multiplied by itself is the term number for that term in the sequence. Therefore we are able to say $t_n = n \times n$, where n is the term number.

Fibonacci Sequence

The **Fibonacci sequence** is a sequence with the first numbers being 0 and 1 and the following numbers being the sum of the previous two numbers. The following are the first 10 terms of the Fibonacci sequence.

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...\}.$$

Thus we are able to see every term, except for the first and second term, is the sum of the first two terms.

ie. $2 = 1 + 1$, $3 = 2 + 1$

The sequence can be represented by the formula below:

t_n = previous term + second previous term, where $t_1 = 0, t_2 = 1$, and n is the term number.
$t_n = t_{n-1} + t_{n-2}$, where $t_1 = 0, t_2 = 1$, and n is the term number.

Problem Set

1. Find the 7th and 8th terms using the method done in class.
   a) $\{4, 12, 20, 28, 36, ...\}$ b) $\{293, 278, 263, 248, 233, ...\}$ c) $\{1, 2, 5, 10, 17, ...\}$

2. Are the following arithmetic sequences, geometric sequences, or neither? If the sequence is arithmetic find the common difference. If it is a geometric sequence find the common ratio.
   a) $\{500, 475, 450, 425, 400, ...\}$ b) $\{2, 14, 98, 686, 4802, ...\}$ c) $\{16384, 4096, 1024, 256, 64, ...\}$

3. Find the 8th and 9th term for the following sequences.
   a) $\{7, 48, 89, 130, 171, ...\}$ b) $\{\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, ...\}$ c) $\{165, 156, 147, 138, 129, ...\}$

4. Liam made three cookies in April, 9 cookies in May, 27 cookies in June, 81 cookies in July, and 243 cookies in August. Is this an arithmetic sequence or geometric sequence?

5. What is the missing number in the sequence?

$$\{15, 14, 29, \square, 72, 115, 187, ...\}$$

6. What is the missing number in the sequence?

$$\{1, 5, 14, \square, 55, 91, 140\}$$

7. What is the missing number in the sequence?

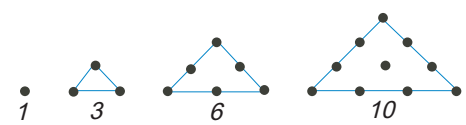
$$\{415, 257, 158, 99, 59, 40, \square\}$$

8. Becca ran for 11 minutes on Monday, 27 minutes on Tuesday, 43 minutes on Wednesday, 59 minutes on Thursday, and 75 minutes on Friday. If Becca continues this pattern, how many minutes will Becca run on Saturday?

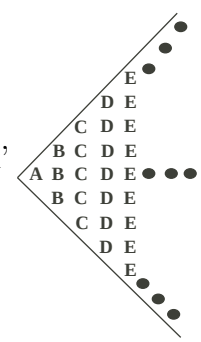
9. What is the sum of the first 6 square numbers?

10. The terms in the sequence $\{2, 7, 12, 17, 22, ...\}$ increases by fives. The terms in the sequence $\{3, 10, 17, 24, 31, ...\}$ increase by sevens. The number 17 occurs in **both** sequences. What is the next number that occurs in **both** sequences?

11. The first four triangular numbers are 1, 3, 6, and 10. What is the seventh triangular number?



12. If the sequence to the right continues, how many letters will be in the 'k' column?



13. What is the next number in the following sequence?

$$\{1, 3, 7, 15, 31, ...\}$$

14. If the pattern □●△○■ repeats, what would be the 214th shape be?

15. Starting at 777 and counting backwards by 5's, Heather counts 777, 772, 767, etc. What is the last number Heather will count that is greater than one?

16. Each number in a sequence is obtained by adding the two previous numbers, like the Fibonacci sequence. The 6th, 7th, and 8th numbers of the sequence are 29, 47, and 76. What is the 3rd number in the sequence?

17. Kolby decides she is going to hop from her house to her friend's house down the street. However, she hops in a pattern taking two hops forward and one hop backwards. If her friend's house is 15 hops away from her, what is the least number of hops she must take to reach her friend's house?

18. What is the sum of all odd integers between 1 and 99 inclusive?

Problem Set Solutions

1.
   a) $t_7 = 52, t_8 = 60$
   b) $t_7 = 203, t_8 = 188$
   c) $t_7 = 37, t_8 = 50$

2.
   a) Arithmetic
      Common difference $= 25$
   b) Geometric
      Common ratio $= 7$
   c) Geometric
      Common ratio $= \frac{1}{4}$

3.
   a) $t_8 = 294, t_9 = 335$
   b) $t_8 = \frac{8}{3}, t_9 = 3$
   c) $t_8 = 102, t_9 = 93$

4. Geometric

5. 43

6. 30

7. 19

8. 91 minutes

9. 91

10. 52

11. 28

12. 21

13. 63

14. ◯

15. 2

16. 7

17. 38

18. 2500