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Senior Math Circles
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Counting Techniques and Probability II

Counting Rules

Multiplication Rule - If I perform a task in m ways AND a second, unrelated task in n ways then I perform both tasks in mn ways. For example, in a 21 speed bike there are 3 ways to select the front sprocket and 7 ways to select the rear.

Addition Rule - If I perform a task in m ways and a second task in n ways, then I can perform either task 1 OR task 2 (but not both) in $m+n$ ways. For example, an instructor can select a person from a class of 12 males and 7 females in $12 + 7 = 19$ ways.

Permutations

In permutations order matters, and objects are selected without replacement.

Example:
Suppose we select 10 objects from a bag (containing 10 objects) and place them in a line.
Let each position be denoted by "__".
Then, there are 10 positions:

__ __ __ __ __ __ __ __ __ __

We fill the first position with any of the 10 objects in the bag:
_10_ __ __ __ __ __ __ __ __ __

Since we did not replace the object, we have 9 objects left in the bag, hence:
_10_ _9_ __ __ __ __ __ __ __ __

And so on, until:
_10_ _9_ _8_ _7_ _6_ _5_ _4_ _3_ _2_ _1_

In terms of counting the number of ways we can select (order) our 10 objects from the bag, we use the multiplication rule. We can select the first object in 10 ways, the next object in 9 ways, the next object in 8 ways .. and the last object in 1 way. In other words, $(10)(9)(8)...(2)(1) = 3,628,800$ ways.

Factorials

$n! = n(n-1)(n-2)...(2)(1)$

We define:
$0! = 1$
$1! = 1$

Tricks:

1. Write down your exceptions first.
2. Draw boxes and fill them in.

Examples:
(Without an exception) In how many ways can I arrange the letters of the word MATH?
$\underline{\ 4\ }\quad \underline{\ 3\ }\quad \underline{\ 2\ }\quad \underline{\ 1\ } = 4!$

(With an exception) In how many ways can I arrange the letters of the word MATH where M is the first letter?
$\underline{\ \text{M}\ }\quad \underline{\ 3\ }\quad \underline{\ 2\ }\quad \underline{\ 1\ } = 3!$

Duplicates

Duplicate objects can be uncounted by dividing by the arrangements of the duplicates.

Example:
In how many ways can I arrange the letters of the word APPLIED?
$\underline{\ 7\ }\quad \underline{\ 6\ }\quad \underline{\ 5\ }\quad \underline{\ 4\ }\quad \underline{\ 3\ }\quad \underline{\ 2\ }\quad \underline{\ 1\ } = 7!$
Notice that there are 2 Ps in APPLIED and there are $2 \times 1 = 2!$ ways to arrange them.
Since both arrangements will look the same we divide by 2!, so there are $\frac{7!}{2!}$ arrangements.

Combinations

In combinations, order does not matter and objects are drawn without replacement.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This number is the number of ways to choose r objects from a set of n objects.

Notes:

1. $\binom{n}{1} = n$
2. $\binom{n}{n} = 1$

3. $\binom{n}{0} = 1$

Example:
In how many ways can I choose 4 letters from the word CHARMED?
$\binom{7}{4} = \frac{7!}{4!(7-4)!} = 35$
There are 35 ways.

Probability Definitions

Recall: The relative frequency definition of probability states that the probability of event A is the number of times event A occurred, denoted by $|A|$, in n trials. Hence $Pr(A) = \frac{|A|}{n}$.

Let $|E|$ be the number of outcomes in event E.
Let $|S|$ be the number of outcomes in sample space S.
Then the mathematical definition of probability states that $Pr(E) = \frac{|E|}{|S|}$.

The probability function is $f(x) = Pr(X = x)$.
It has properties:

1. $0 \leq f(x) \leq 1$
2. $\sum f(x) = 1$

Mutually Exclusive

Two events E and F are mutually exclusive (disjoint) if and only if $Pr(E \text{ and } F) = 0$.
Note: Another way of writing $Pr(E \text{ and } F)$ is $Pr(EF)$.

Independence

Two events E and F are independent if and only if $Pr(E \text{ and } F) = Pr(EF) = Pr(E)Pr(F)$.

Probabilities for Unions

The probability of event E or event F or both is given by:
$Pr(E \text{ or } F) = Pr(E) + Pr(F) - Pr(EF)$

Example:
The probability that a person does well on a test is 0.3. The probability that a person sleeps the night before is 0.5. The probability that a person sleeps the night before a test and does well is 0.15. Is doing well independent of your sleep the night before?

Probably - but this is a made up example. In terms of our definition, let E be the event that you slept the night before. Let F be the event that you do well.
$Pr(E)Pr(F) = (0.3)(0.5) = 0.15$
Since this equals $Pr(EF)$, doing well on a test is independent of your sleep the night before.

The point is that independence is a statistical/mathematical concept - not a concept that you can argue.

Example:
A student is sleepy with probability 0.6 and grumpy with probability 0.3. The probability that a student is both sleepy and grumpy is 0.25.

a) Is the event sleepy independent of grumpy?

b) Is the event sleepy disjoint from grumpy?

Solution:

a) $Pr(\text{sleepy})Pr(\text{grumpy}) = 0.3(0.6) = 0.18$, BUT $Pr(\text{sleepy and grumpy}) = 0.25$. These are not equal so they are not independent.

b) $Pr(\text{sleepy and grumpy}) > 0$, hence they are not disjoint.

Problem Set

1. A bridge hand has 13 cards (picked at random from a deck of 52).
   (a) What is $|S|$?
   (b) What is the probability that my hand contains:
      i. 3 aces
      ii. at least 1 ace
      iii. 6 spades, 4 hearts, 2 diamonds and 1 club
      iv. a 6-4-2-1 split between the 4 suits
      v. a 5-4-2-2 split between the 4 suits
2. The letters of the word STATISTICS are arranged at random.
   (a) What is $|S|$?
   (b) What is the probability that:
      i. the letter S occurs on both ends?
      ii. the same letter occurs at both ends?
      iii. the letters A and C occur together, in the order CA?
      iv. the letters A and C occur together, in any order?
3. In how many ways can the word MATH be arranged in a circle?
4. The events A and B are defined by $Pr(A) = Pr(B) = Pr(AB) = 1$. The events A and B are:
   A) independent and disjoint
   B) independent and not disjoint
   C) not independent nor disjoint
   D) mutually exclusive and not independent
   E) none of the above
5. Let event A be selecting a person with brown eyes from a small population without replacement and the event B be selecting a second with brown eyes. These events are:
   A) independent and not disjoint
   B) mututally exclusive and not independent
   C) independent and disjoint
   D) not independent nor disjoint
   E) none of the above
6. What is the probability that, in selecting two cards, we get 2 hearts in a row:
   a) if we sample with replacement?

b) if we sample without replacement?

7. A random number generator on a computer can give a sequence of independent random digits chosen from $S = \{0, ..., 9\}$. This means that each digit has a probability of $\frac{1}{10}$ of being any of $\{0, ..., 9\}$ and the outcomes for the different trials are independent of one another. Determine the probability that:

   a) in a sequence of 5 trials, all the digits generated are odd.

   b) the number 9 occus for the first time on trial 10.

Answers

1. (a) $\binom{52}{13}$

   (b) i. $\dfrac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}}$

   ii. $\dfrac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} + \dfrac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} + \dfrac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}} + \dfrac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}}$

   iii. $\dfrac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}}$

   iv. $4!\dfrac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}}$

   v. $\dfrac{4!}{2!} \times \dfrac{\binom{13}{5}\binom{13}{4}\binom{13}{2}\binom{13}{2}}{\binom{52}{13}}$

2. (a) $|S| = \dfrac{10!}{3!3!2!}$

   (b) i. __S__ all other letters __S__ $= |E| = \dfrac{8!}{3!2!}$. The probability is $\dfrac{|E|}{|S|} = \dfrac{1}{15}$.

   ii. An I on either end will give $|F| = \dfrac{8!}{3!3!}$. A T on either end would give $|H| = \dfrac{8!}{3!2!}$. Hence, our probability is $\dfrac{|F| + |H| + |E|}{|S|} = \dfrac{7}{45}$.

   iii. We have 9 objects: S, T, I, CA. Hence our probability is $\dfrac{9!}{3!3!2!} \div |S|$.

   iv. We can arrange CA in two ways, hence our probability is $\dfrac{9!}{3!3!} \div |S|$.

3. Hold the M in place, and rearrange the others giving a probability of 3!.

4. B

5. D

6. a) $\left(\dfrac{13}{52}\right)^2$

b) $\left(\frac{13}{52}\right)\left(\frac{12}{51}\right)$

7. a) $\left(\frac{1}{2}\right)^5$

b) $\left(\frac{9}{10}\right)^9\left(\frac{1}{10}\right)$