University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

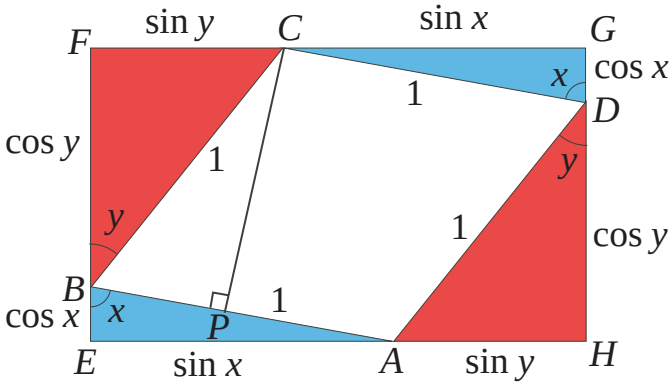
Senior Math Circles
November 12, 2008
Trigonometry III

Trigonometry

Sum of Angles

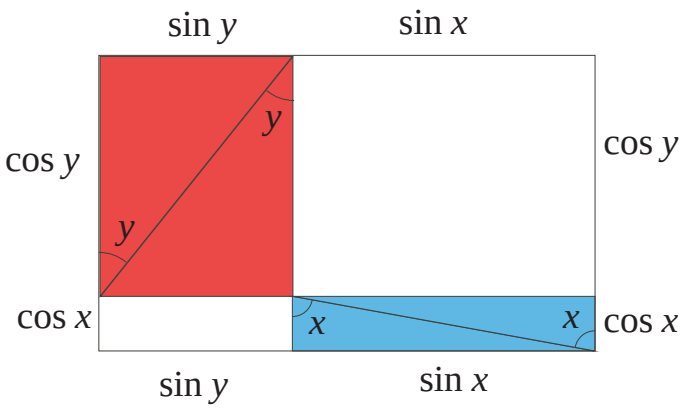
Theorem: $\sin(x+y) = \sin x \cos y + \cos x \sin y$

Visual Proof:
In the diagram below, we have four right triangles, each with a hypotenuse of length 1. Two of the triangles have an acute angle x and the other two have an acute angle y.



Now, consider the white area in the middle. It is a rhombus since $AB = BC = CD = DA = 1$, so if we draw a line CP such that CP meets AB at a right angle, the area of the white region is $CP \times AB$.
In $\triangle CBP$, $\angle CBP = 180^\circ - x - y$, and $\sin(\angle CBP) = \frac{CP}{1}$. Therefore, $CP = \sin(180^\circ - x - y) = \sin(x+y)$.
Therefore, the area of the white region is $CP \times AB = \sin(x+y)$.

Now, if we translate the bottom right triangle up $\cos x$ and left $\sin x$, it will form a rectangle with the upper left triangle. We then translate the bottom left triangle right $\sin y$, and the top right triangle down $\cos y$ so that they too form a rectangle. From these translations we get the diagram below.



As you can see, the dimensions of the large rectangle are the same, and so its area is the same. Also, the triangles have only been translated and are not overlapping, so the sum of their areas is the same. Therefore, the area of the white region must be the same.
Each of the white areas are rectangles, and so the sum of their areas is $\sin x \cos y + \cos x \sin y$.

Equating the areas of the white regions in each diagram, we get
$\sin(x + y) = \sin x \cos y + \cos x \sin y$.

Now, we can derive the formula $\sin(x - y) = \sin x \cos y - \cos x \sin y$.

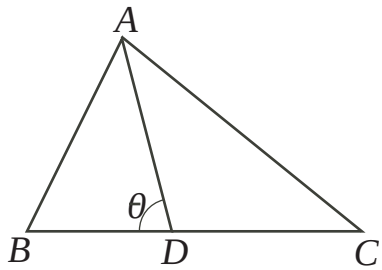
We know that $\sin(a + b) = \sin a \cos b + \cos a \sin b$
for all angles a and b. Now, let $a = x$ and let $b = -y$.
We have $\sin(x + (-y)) = \sin x \cos(-y) + \cos x \sin(-y)$, and since
$\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, this gives us
$\sin(x - y) = \sin x \cos y + \cos x(-\sin y) = \sin x \cos y - \cos x \sin y$.

Solving Triangles

Theorem (Sine Law):
In a triangle ABC with angles A, B and C and side lengths a, b and c opposite those angles, respectively, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Theorem (Cosine Law):
In a triangle ABC with angles A, B and C and side lengths a, b and c opposite those angles, respectively, $c^2 = a^2 + b^2 - 2ab\cos C$.

Example:



In the triangle above, $AB = 6, AD = 5, AC = \sqrt{27}$ and $BD = 2$.
Find DC.

Solution:
In $\triangle ABD$,

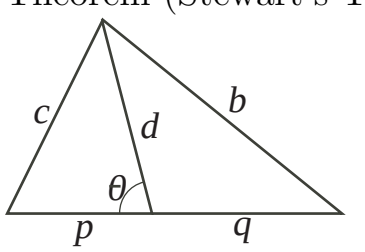
$$\begin{aligned} 6^2 &= 5^2 + 2^2 - 2(5)(2)\cos\theta \\ 36 &= 29 - 20\cos\theta \\ 7 &= -20\cos\theta \\ \tfrac{-7}{20} &= \cos\theta \end{aligned}$$

Now, in $\triangle ADC$,

$$\begin{aligned} (\sqrt{27})^2 &= 5^2 + x^2 - 2(5)(x)\cos(180^\circ - \theta) \\ 27 &= 25 + x^2 - 10x(-\cos\theta) \\ 0 &= x^2 + \tfrac{7}{2}x - 2 \\ 0 &= 2x^2 + 7x - 4 \end{aligned}$$

Since $x > 0$, $x = 4$.

Theorem (Stewart's Theorem):



$pb^2 + qc^2 = (p+q)d^2 + pq^2 + qp^2$

Proof:
In the left side triangle, using the cosine law gives us
$c^2 = p^2 + d^2 - 2pd\cos\theta$.
In the right side triangle, using the cosine law gives us
$b^2 = q^2 + d^2 - 2qd\cos(180^\circ - \theta)$
$b^2 = q^2 + d^2 + 2qd\cos\theta$.

Now multiplying the first equation by q gives us
$qc^2 = qp^2 + qd^2 - 2pqd\cos\theta$,
and multiplying the second equation by p gives us
$pb^2 = pq^2 + pd^2 + 2pqd\cos\theta$.

When we add these two equations together we get
$pb^2 + qc^2 = (p+q)d^2 + pq^2 + qp^2$.

Problem Set

1. What is $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$?

2. In triangle ABC, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D, and $CD = 8$. What is BD?

3. $ABCD$ is a quadrilateral such that $BC = 4$, $AD = 5$, $AB = 6$, and $BD = 7$. If $\angle A+\angle C = 180°$, what is CD?

4. Points K, L, M, and N lie in the plane of the square $ABCD$, but outside the square, so that AKB, BLC, CMD, and DNA are equilateral triangles. If $ABCD$ has an area of 16, what is the area of $KLMN$?

5. Triangle ABC is a right triangle with $\angle ACB$ as its right angle, $\angle ABC = 60°$, and $AB = 10$. Let P be randomly chosen inside ABC, and extend BP to meet AC at D. What is the probability that $BD > 5\sqrt{2}$?

6. Points P and Q are located inside the square $ABCD$ such that DP is parallel to QB, and $DP = QB = PQ$. Determine the minimum possible value of $\angle ADP$.

7. In triangle ABC, $\angle ABC = 45°$. Point D is on BC, between B and C, so that $2BD = CD$ and $\angle DAB = 15°$. What is $\angle ACB$?

8. What is $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$?