University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Senior Math Circles
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Conics II

Locus Problems

The word "locus" is sometimes synonymous with "set" e.g. "the locus of points equidistant from d and F."
However, in math problems you typically see something more like the following:

- fixed points are given
- one variable point V, lying on a curve, is given
- then a construction yields another point X
- you are asked to find "the locus of X as V varies"

Sometimes the locus can be hideous, but on contest problems it is almost always a line, a circle, or some set that can be nicely described in lines and circles.

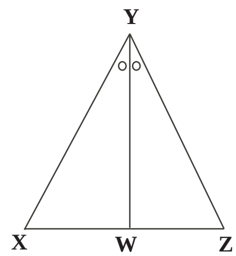
Circles of Apollonius

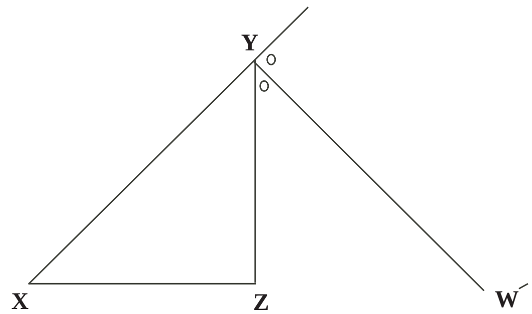
Let A, B be two points and $k > 1$ a fixed constant.
Theorem: the locus $\{P : \|AP\| = k\|BP\|\}$ is a circle.

Proof 1 (co-ordinates): let $A = (0, 0)$ and $B = (1, 0)$ without loss of generality (e.g. by a similarity transformation).
$\|AP\| = k\|BP\| \iff \sqrt{x^2 + y^2} = k\sqrt{(x-1)^2 + y^2}$
$\iff x^2 + y^2 = k[(x^2 - 2x + 1) + y^2]$
$\iff (k-1)x^2 + (k-1)y^2 - 2kx + k = 0$ which by completing the square is of the form
$(k-1)(x - x_0)^2 + (k-1)y^2 = \text{constant}.$
Thus the locus is empty, a point, or a circle. Now observe that the locus is nonempty and contains at least 2 distinct points (e.g. we can explicitly compute two such points on the line AB).$\square$

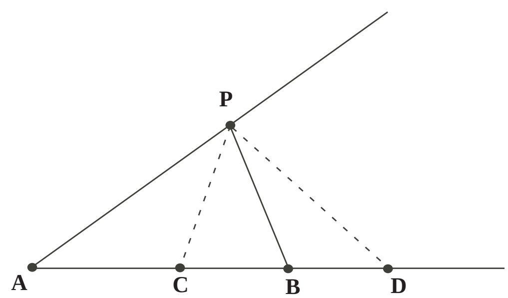
Proof 2 (analytic geometry):

Lemma: If XYZ is a triangle and W is the point on XZ so that YW bisects $\angle XYZ$, then $\frac{\|YX\|}{\|YZ\|} = \frac{\|WX\|}{\|WZ\|}$
Proof: (use the sine law)
Similarly: If W' bisects $\angle XYZ$ externally and W' lies on XZ extended, then $\frac{\|YX\|}{\|YZ\|} = \frac{\|W'X\|}{\|W'Z\|}$





Back to main proof...
Let P be a point on the locus, and let C, D be the internal & external bisectors of $\angle APB$. Note by the lemmas that $\frac{\|AC\|}{\|CB\|} = k$ and $\frac{\|AD\|}{\|BD\|} = k$. So actually C, D are independent of P.



Exercise: Show $\angle CPD = 90°$

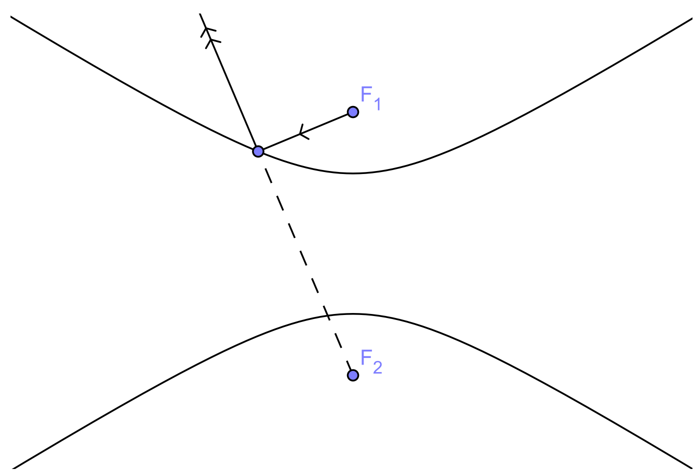
This shows that all points on the locus lie on the circle with diameter CD
To show that the locus equals the circle (instead of a subset of the circle) needs a little more work.
Given any ray R originating at B, if we consider a variable point P originating at B and moving along R, note that $\frac{\|AP\|}{\|BP\|}$ starts at $+\infty$ and tends towards 1, so the locus must meet the ray. This shows the locus cannot miss any points on the circle. $\square$

Hyperbolas

Definition: given two points F_1 and F_2, and a focal radius a, we define a hyperbola to be the locus
$\{P : \|PF_1\| - \|PF_2\| = \pm a\}$

Exercise: If $F_1 = (0, -1)$ and $F_2 = (0, 1)$, find the equation of the locus.

Reflection Property: If a ray leaves one focus, after reflecting, it travels in a line which, extended, passes through the other focus.



Symmetry: The hyperbola has two perpendicular axes of symmetry.

Asymptotes: Each hyperbola has two lines which it gets closer to, but never meets.

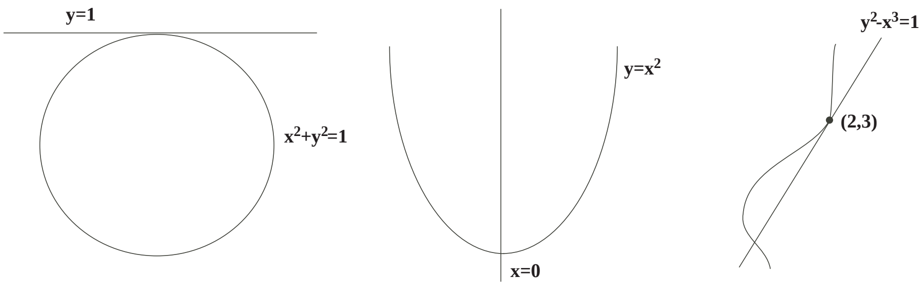
The equation of a <u>rotation</u> clockwise by θ is the affine transformation
$(x, y) \longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

Exercise: Find the image of the hyperbola $\{xy = 1\}$ under a 45° counterclockwise rotation.

Tangents

A line which "just touches" a curve at a point (i.e. has a "double root") is a tangent.

Generally a tangent hits a curve only once but this is not truly necessary or sufficient:



tangent, meets once | non tangent, meets once | tangent at (2,3), meets more than once

Precise definition: Let P be a point on curve C. Let $Q_1, Q_2, \dots$ be a sequence of points on C whose limit is P. For each i let L_i be the line through P and Q_i. The tangent to C at P is $\lim_{i\to\infty} L_i$.

For a conic C, you can show that line L is tangent to C if and only if the equation of their intersection has a double root.

Exercise: Find the equation of the tangent to the parabola $y^2 = 8x$ at $(2,4)$.

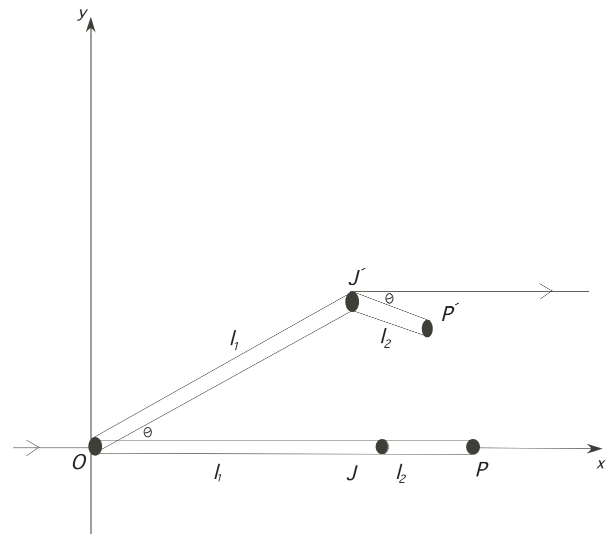
Exercise: Verify that a horizontal ray incoming to $(2,4)$ will pass through the focus $(2,0)$.

Problem Set

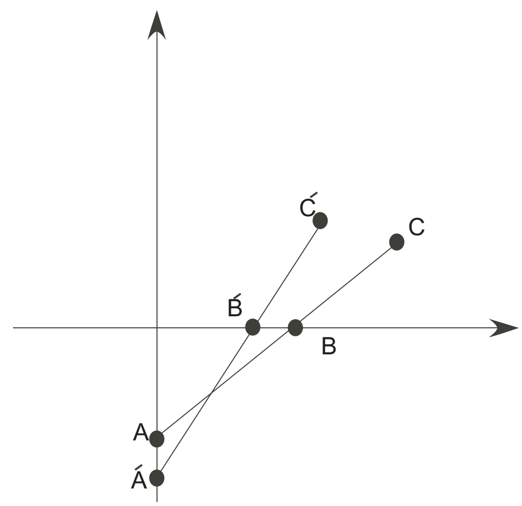
1. Let P_1 be a parabola with focus F and directrix d_1, and P_2 be a parabola with focus F and directrix d_2. Show that every intersection point of P_1 and P_2 lies on an angle bisector of d_1 and d_2.

2. Drawing ellipses:

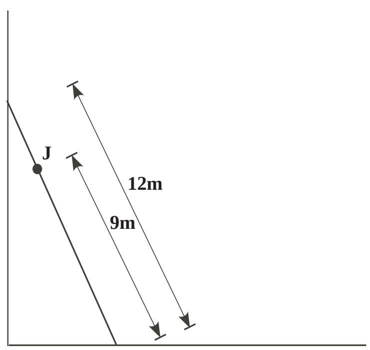
   (a) The spirograph method for drawing an ellipse can be described as follows. We construct a hinged rod; one end is fixed at the origin O. A rod of length l_1, initially pointing right, has O as one endpoint and a joint J as its other endpoint. A second rod of length l_2, initially pointing right, has J as one endpoint and a pen P as its other endpoint. Now rotate the first rod counterclockwise around the origin, and rotate the second rod about the joint clockwise at the same speed. Show the locus of P, as a full rotation occurs, is an ellipse.



(b) In an "ellipsograph", we use as rigid rod with three points A, B, C on it. We use a peg so that A can slide freely along the x-axis, B can slide freely along the y-axis, and a pen is in C. Show the locus of C, as the points A and B slide, is an ellipse.

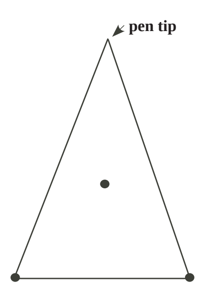


(c) Johnny is standing on a 12m ladder, initially vertical. His feet are 9m from the bottom of the ladder. The ladder starts to slip, but the top end remains in contact with the wall, and the bottom end remains on the ground.



What is the locus of the location of his feet as his ladder slides smoothly to a horizontal position?

3. There are three pegs in the plane which form the vertices of an equilateral triangle with side length 1cm. A string of length 5cm is placed around the pegs. If you pull the loop taut with a pen and go around the pegs once,what curve do you trace out?



4. For this problem use the following special case of "*Bézout's Theorem*": Any two non-degenerate conics intersect in at most 4 points.

   (a) A regular n-gon is inscribed in an ellipse E which is not a circle. What are the possible values of n?

   (b) The term "degenerate conic" means one whose equation is of the form $(ax+by+c)(a'x+b'y+c')=0$. Show that, given any 5 points of which no four are colinear, there is a unique conic containing those 5 points. What if 4 of these points are colinear?

5. Let l be a fixed line, and A, B are two fixed points not on l. Given a point P on l, Q is the foot of the perpendicular from B to AP. What is the locus of Q as P varies over l?

6. C is a circle with center O, and A is a point different from O. Given a point P on C, let M be the intersection of AP with the bisector of $\angle AOP$. What is the locus of M as P ranges over the circle?

Sources:
• "Relations" textbook by Del Grande & Egsgard, Gage, 1979
• "Ye Olde Geometry Shoppe Part II" by J.P Grossman in Volume 6, Issue 3 of Mathematical Mayhem, p. 7-12.