Continued Fractions: Invisible Patterns

Ramana Andra

Thomas Jackson

FTLOMACS
OCT 14, 2017

AGENDA

- Representations on Numbers
- Continued Fraction Notation
- Finite Ratio Example
- Convergents
- Radicals
- Algebraic Numbers
- π
- Java Programs
- Invisible Connections

TRADITIONAL REPRESENTATIONS OF NUMBERS

Infinite Series, Periodic and Nonperiodic Decimal Expansions, Integrals

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} - \frac{1}{13} \ldots$$

$$\int_{-\infty}^{\infty} \sqrt{1 - x^2}\, dx = \frac{\pi}{2}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

3.14159265358979323846264338327 95…….

TRADITIONAL REPRESENTATIONS OF NUMBERS

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots$$

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1\bullet 2} + \frac{1}{1\bullet 2\bullet 3} + \frac{1}{1\bullet 2\bullet 3\bullet 4} + \cdots$$

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718281828459\ldots$$

$$\sqrt{2} = 1.41421356237\ldots$$

EXAMPLES OF CF'S

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cfrac{9^2}{2 + \cfrac{11^2}{2 + \dots}}}}}}$$

EXAMPLES OF CF'S

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \dots}}}}}}}}$$

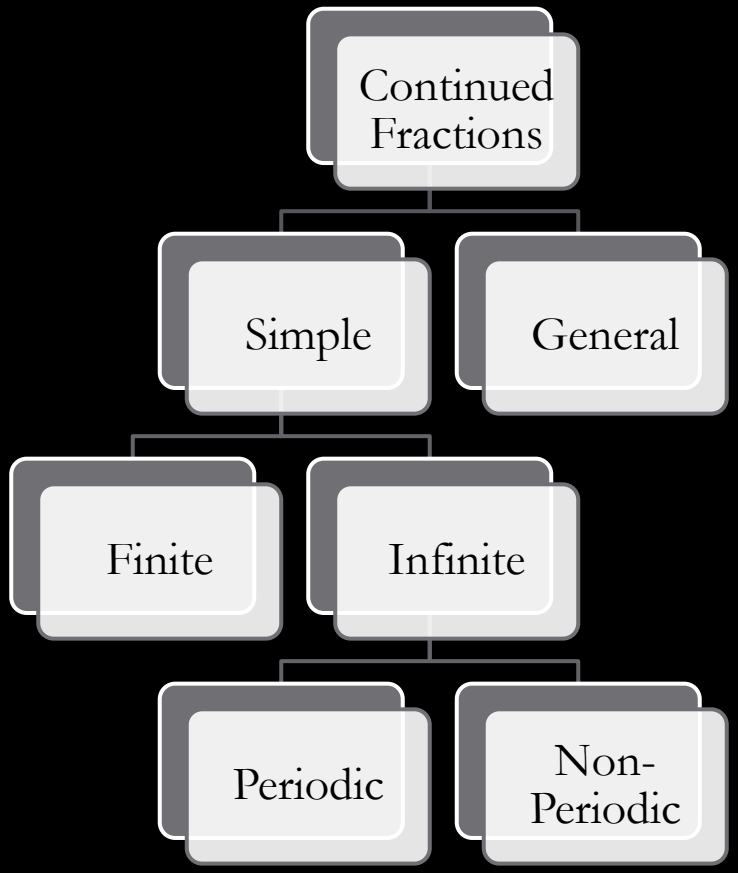
$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}}}$$

EXAMPLES OF CF'S

$$\frac{\pi}{2} = 1 - \cfrac{1}{3 - \cfrac{2\bullet 3}{1 - \cfrac{1\bullet 2}{3 - \cfrac{4\bullet 5}{1 - \cfrac{3\bullet 4}{3 - \cfrac{6\bullet 7}{1 - \cfrac{5\bullet 6}{3 - \dots}}}}}}}$$

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}}$$

CONTINUED FRACTIONS TYPES



CONTINUED FRACTIONS FORMS

General Continued Fraction

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cfrac{b_4}{a_5 + \cfrac{b_5}{a_6 \ldots}}}}}$$

Simple Continued Fraction ($b_i = 1$)

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 \ldots}}}}} = [a_1; a_2, a_3, a_4, \ldots] = [a_1: a_2, a_3, a_4, \ldots]$$

Convergents

$$\frac{p_i}{q_i} = [a_1; a_2\ a_3, a_4, a_5, \ldots a_n]$$

FAMOUS CONTINUED FRACTIONS

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots]$$

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$$

$$\varphi = [1; 1, 1, 1, 1, 1, 1, \ldots]$$

FINITE RATIO EXAMPLE

Consider the fraction $\frac{47}{13}$. Can we find the continued fraction (cf.) for this finite ratio?

We should consider breaking up the fraction into mixed form.

$$\text{So } \frac{47}{13} = 3 + \frac{8}{13}$$

We need the numerator to be one for our cf. form so we can apply the following algebraic identity $\frac{a}{b} = \frac{1}{\frac{b}{a}}$ to move forward.

FINITE RATIO EXAMPLE

$$\frac{47}{13} = 3 + \frac{1}{\frac{13}{8}} = 3 + \frac{1}{1 + \frac{5}{8}} = 3 + \frac{1}{1 + \frac{1}{\frac{8}{5}}} = = 3 + \frac{1}{1 + \frac{1}{1 + \frac{3}{5}}}$$

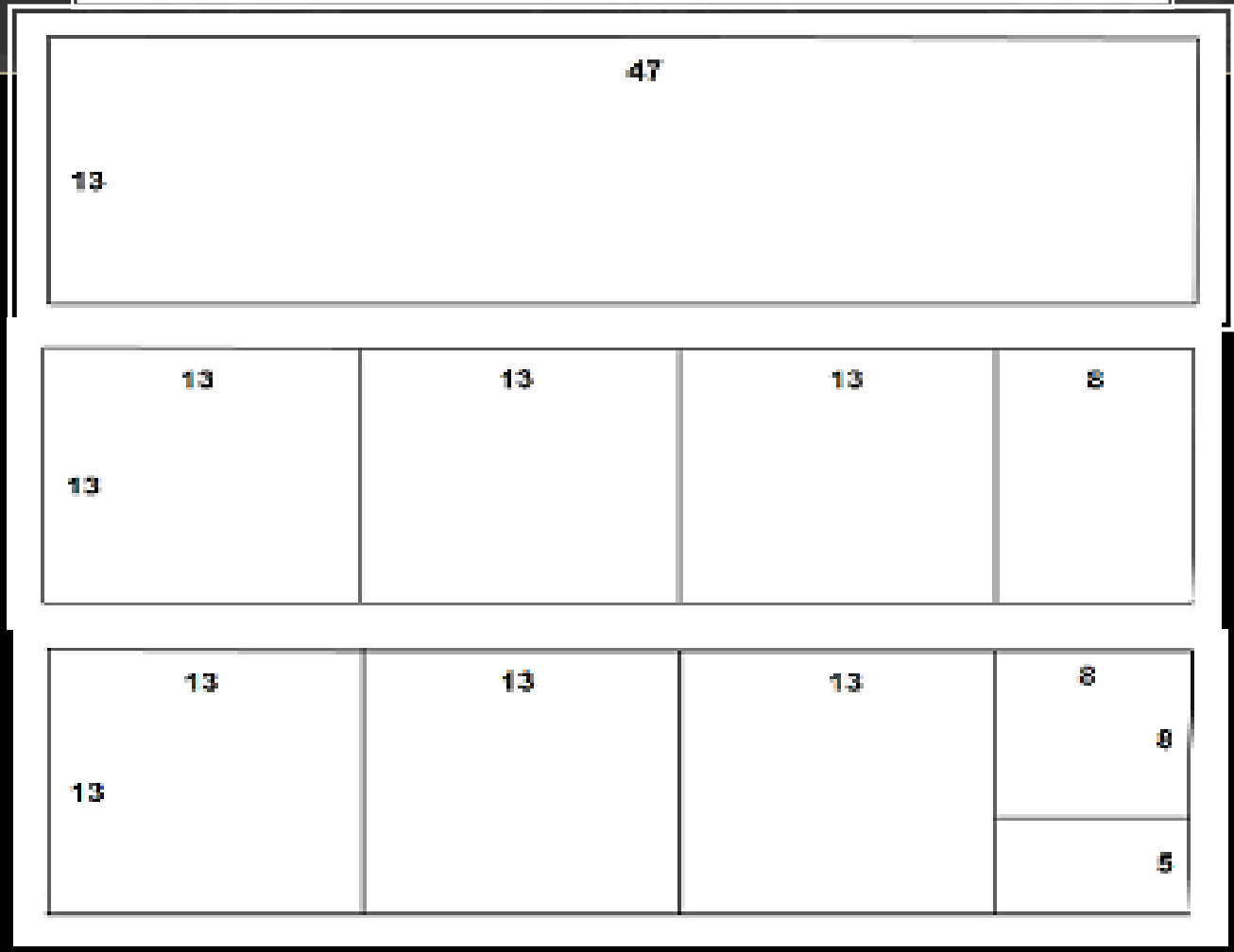
$$\frac{47}{13} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{5}{3}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{2}{3}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{2}{3}}}}$$

$$\frac{47}{13} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\frac{3}{2}}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}}$$

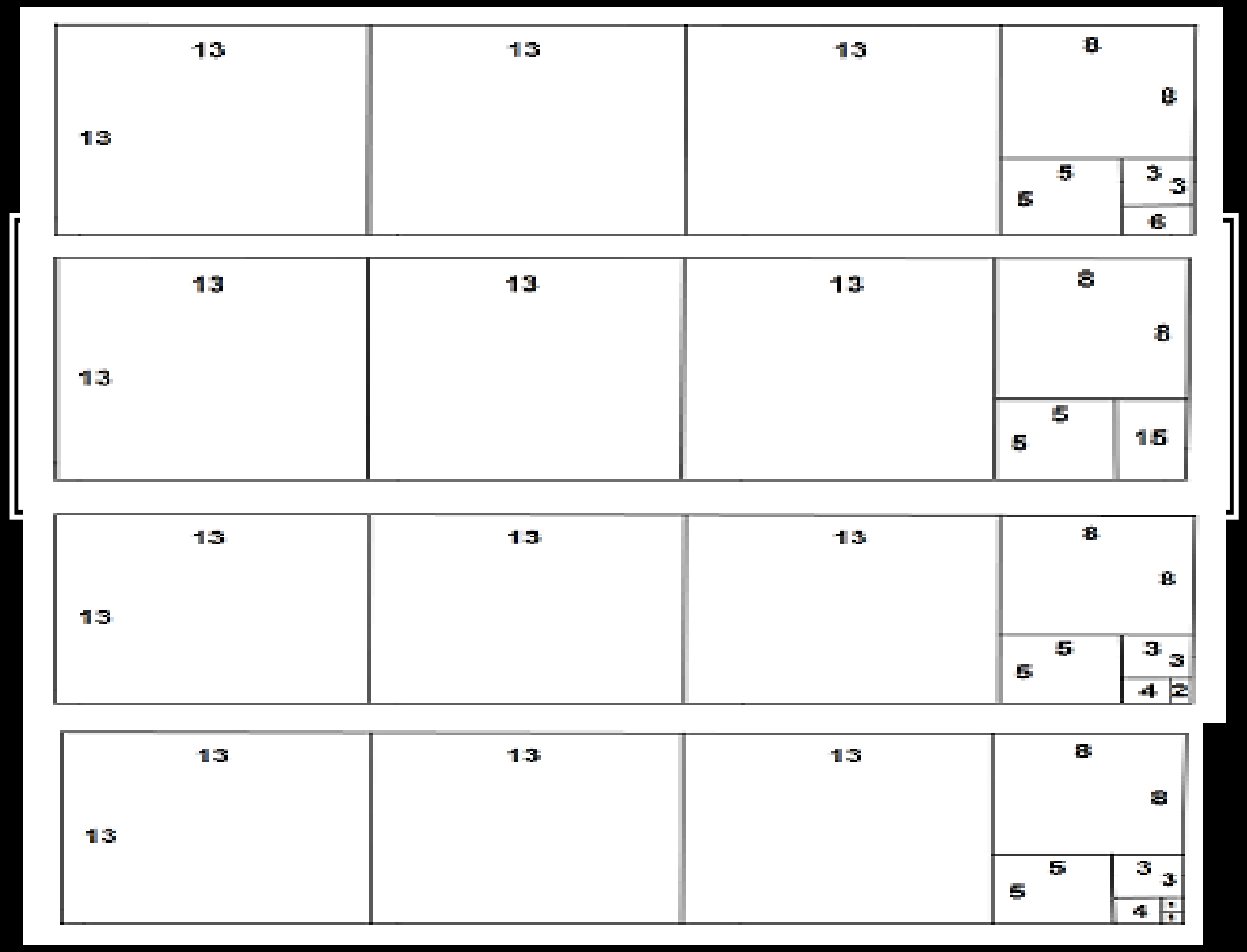
FINITE RATIO EXAMPLE

This procedure is an example of an algorithm called the Euclidean Algorithm which was developed by Euclid in his famous book The Elements. We can also look at the problem from geometrical and coding perspectives.

GEOMETRIC CONSTRUCTION



Geometric Construction



EUCLIDEAN ALGORITHM

Let's consider the calculations already performed for the fraction $\frac{47}{13}$.

$$47 = \mathbf{3}(13) + 8$$
$$13 = \mathbf{1}(8) + 5$$
$$8 = \mathbf{1}(5) + 3$$
$$5 = \mathbf{1}(3) + 2$$
$$3 = \mathbf{1}(2) + 1$$
$$2 = \mathbf{2}(1) + 0$$

EUCLIDEAN ALGORITHM

We continue this operation on the two numbers and it will always stop when the length of the shortest rectangle is one and the remainder after the last triangle has been divided is zero.

We can find the cf. form from the leading numbers from the list:

$$\frac{47}{13} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}}}$$

PHI DERIVATION

We all know that ϕ is derived from the roots of the quadratic equation

$$\varphi^2 - \phi - 1 = 0$$

$$\varphi = 1 + \frac{1}{\varphi}$$

$$\varphi = 1 + \frac{1}{1 + \frac{1}{\varphi}}$$

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\varphi}}}$$

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}}$$

$$\varphi = [1; 1, 1, 1, 1, 1, 1, 1, \ldots]$$

CONVERGENTS

We can look at the partial cfs for the fraction $\frac{47}{13}$ to find better approximations to the exact value of this fraction. These values are called convergents.

$$3,\ 3+\frac{1}{1},\ 3+\frac{1}{1+\frac{1}{1}},\ 3+\frac{1}{1+\frac{1}{1+\frac{1}{1}}},\ 3+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}$$

$$3,\ 4,\ \frac{7}{2},\ \frac{11}{3},\ \frac{18}{5},\ \frac{47}{13} = 3,\ 4,\ 3.5,\ 3.666..,\ 3.6,\ \mathbf{3.65\ldots}$$

CONVERGENTS

We can generalize this procedure. Let $c_i = \frac{p_i}{q_i}$ represent the convergents for a given fraction with $i \geq 0$.

$$\text{So } c_1 = \frac{p_1}{q_1} = a_1 = \frac{a_1}{1}$$

$$c_2 = \frac{p_2}{q_2} = a_1 + \frac{1}{a_2} = \frac{a_1 a_2 + 1}{a_2}$$

$$c_3 = \frac{p_3}{q_3} = a_1 + \frac{1}{a_2 + \frac{1}{a_3}} = \frac{a_1 a_2 a_3 + a_1 + a_3}{a_2 a_3 + 1}$$

CONVERGENTS

$$c_4 = \frac{p_4}{q_4} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}} = \frac{a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1}{a_2a_3a_4 + a_2 + a_4}$$

So we can "see" the following recursions for the convergent terms if we look carefully:

$$p_i = a_i p_{i-1} + p_{i-2}$$

$$q_i = a_i q_{i-1} + q_{i-2}$$

CONVERGENTS

Using these recursion formulas, the convergents for the golden ratio ϕ are

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{18}, \ldots..$$

We can see that the numbers in the numerators and denominators are terms from the Fibonacci sequence!

We should notice that

$$p_i = a_i p_{i-1} + p_{i-2} = (1)p_{i-1} + p_{i-2} = p_{i-1} + p_{i-2}$$

$$q_i = a_i q_{i-1} + q_{i-2} = (1)q_{i-1} + q_{i-2} = q_{i-1} + q_{i-2}$$

and $c_i = \frac{t_{i+1}}{t_i}$ where t_i are the terms of the Fibonacci sequence.

SQUARE ROOT 2

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}} = [1; 2, 2, 2, 2, 2, 2, \ldots]$$

SQUARE ROOT 2

Let's see how we can derive this cf representation.

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{\frac{1}{\sqrt{2}-1}} = 1 + \frac{1}{\sqrt{2}+1}$$

But $\sqrt{2} + 1 = 2 + (\sqrt{2} - 1) = 2 + \frac{1}{\frac{1}{\sqrt{2}-1}} = 2 + \frac{1}{\sqrt{2}+1}$

Thus $\sqrt{2} = 1 + \frac{1}{2+\frac{1}{\sqrt{2}+1}}$

If we use this rationalization method again repeatedly, we find the cf. of $\sqrt{2}$ consists of a single 1 followed entirely by 2's.

Thus $\sqrt{2} = [1; 2, 2, 2, 2, 2, \ldots]$

SQUARE ROOT 17

Is there a pattern in the representation?

Can we generalize numbers of the form $\sqrt{n^2+1}$?

Let's give it a go!

PERIODIC CF'S

Let's consider a simple case and suppose

$$A = [a; b, b, b, b, b, \ldots]$$

We can rewrite A in the form $A = a + \frac{1}{[b; b, b, b, b, b, \ldots]}$

We need to determine the value of $B = [b; b, b, b, b, b, \ldots]$

Just like for A, we can rewrite $B = b + \frac{1}{[b; b, b, b, b, b, \ldots]}$

PERIODIC CF'S

So we have B = b + $\frac{1}{B}$ which can be rewritten as $B^2 - bB - 1 = 0$

Using the Quadratic Formula, we have B = $\frac{b+\sqrt{b^2+4}}{2}$

Thus A = a + $\frac{2}{b+\sqrt{b^2+4}}$ = a - $\frac{b-\sqrt{b^2+4}}{2}$ = $\frac{2a-b}{2} + \frac{\sqrt{b^2+4}}{2}$

So we have $\frac{2a-b}{2} + \frac{\sqrt{b^2+4}}{2}$ = [a; b, b, b, b, …]

If b = 2a, then $\sqrt{a^2+1} = [a; 2a, 2a, 2a, 2a, \ldots]$

SQUARE ROOT 3

Let's try to derive the continued fraction for $\sqrt{3}$ on the board together!

SQUARE ROOT 3

$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}}}$$

$$= [1; 1, 2, 1, 2, 1, 2, \ldots]$$

.

PERIODIC CF'S

D	Continued fraction of $\sqrt{D}$	Period
2	$[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \cdots]$	1
3	$[1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \cdots]$	2
5	$[2, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, \cdots]$	1
7	$[2, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, 1, 1, 4, 1, \cdots]$	4
11	$[3, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, \cdots]$	2
13	$[3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, 1, 1, \cdots]$	5
17	$[4, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, \cdots]$	1
19	$[4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8, \cdots]$	6
23	$[4, 1, 3, 1, 8, 1, 3, 1, 8, 1, 3, 1, 8, 1, 3, 1, 8, 1, \cdots]$	4
29	$[5, 2, 1, 1, 2, 10, 2, 1, 1, 2, 10, 2, 1, 1, 2, 10, 2, 1, 1, \cdots]$	5
31	$[5, 1, 1, 3, 5, 3, 1, 1, 10, 1, 1, 3, 5, 3, 1, 1, 10, 1, 1, \cdots]$	8
37	$[6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, \cdots]$	1

CUBE ROOTS

What about cube roots?

These continued fractions are not periodic.

We will find their cf. representation
from their decimal first.

From the decimal part,
we will repeatedly invert the fractional part.

CUBE ROOTS

$$
\begin{aligned}
\sqrt[3]{2} &= 1.259921\ldots \\
&= 1 + \cfrac{1}{3.847322\ldots} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1.180189\ldots}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5.549736\ldots}}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1.819053\ldots}}}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1.220922\ldots}}}}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4.526491\ldots}}}}}}
\end{aligned}
$$

INFINITE CONTINUED FRACTIONS

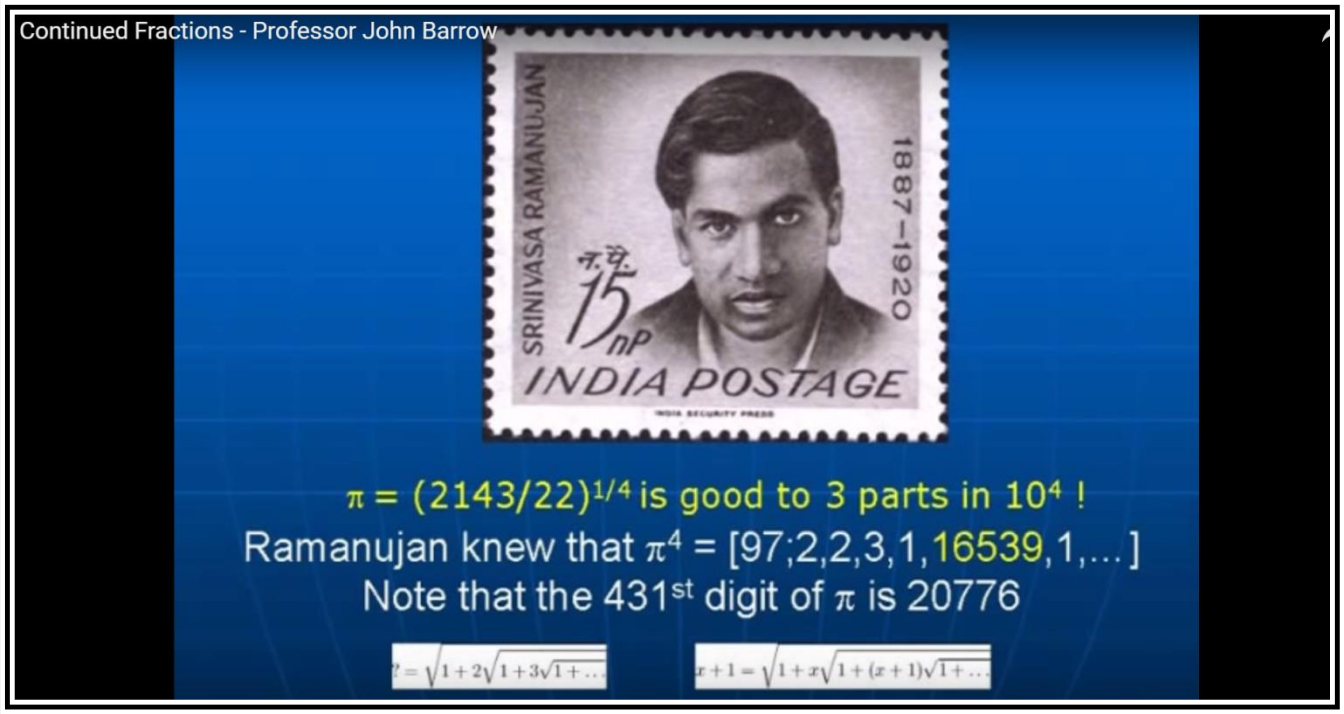
$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots],$$

$$\sqrt[3]{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, \ldots],$$

$$\sqrt{2} = [1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots],$$

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, \ldots].$$

$\Pi^{1/4}$


Continued Fractions - Professor John Barrow
SRINIVASA RAMANUJAN
1887–1920
15 nP
INDIA POSTAGE
π = (2143/22)^1/4 is good to 3 parts in 10^4 !
Ramanujan knew that π^4 = [97;2,2,3,1,16539,1,...]
Note that the 431st digit of π is 20776

Π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + 0.63459101439547237857177\ldots}}}}$$

CUBE ROOTS

We can find the cube roots using the bisection method.

Easy for high school students to understand.

Can be easily coded in a programming language.

We have some simple Java programs to demonstrate.

JAVA PROGRAMS

Rational Numbers

Square Root

Cube Root

π

FORMULA FOR Π

How do we evaluate Pi using a formula?

We use an identity that came up recently in our Math club meeting.

$$\pi = 16 \arctan\left(\frac{1}{5}\right) - 4 \arctan\left(\frac{1}{239}\right)$$

RAMANUJAN

Ramanujan proved two connections between **π, e** and **ϕ:**

$$\left(\sqrt{2+\Phi}-\Phi\right)e^{2\pi/5} = 1 + \cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{1+\dots}}}$$

$$\left(\sqrt{2-\phi}-\phi\right)e^{\pi/5} = 1 - \cfrac{e^{-\pi}}{1+\cfrac{e^{-2\pi}}{1-\cfrac{e^{-3\pi}}{1+\dots}}}$$

A CONTINUED FRACTION APPROXIMATION OF THE GAMMA FUNCTION

$$\Gamma(x+1) \approx \sqrt{2\pi x} \cdot e^{-x} \left(x + \cfrac{1}{12x - \cfrac{1}{10x + \cfrac{a}{x + \cfrac{b}{x + \cfrac{c}{x + \cfrac{d}{x + \ddots}}}}}} \right)^{x},$$

REFERENCES

You Tube video: **Continued Fractions - Professor John Barrow**

https://www.youtube.com/watch?v=zCFF1l7NzVQ&t=670s

The Topsy-Turvy World of Continued Fractions:

https://www.math.brown.edu/~jhs/frintonlinechapters.pdf

Cube Root of a Number:

http://www.geeksforgeeks.org/find-cubic-root-of-a-number/

Gamma Function:

http://www.sciencedirect.com/science/article/pii/S0022247X12009274

A Continued Fraction Approximation of the Gamma Function:

http://www.sciencedirect.com/science/article/pii/S0022247X12009274

https://ac.els-cdn.com/S0022247X12009274/1-s2.0-S0022247X12009274-main.pdf?_tid=4981f0e4-b07a-11e7-bb6a-00000aab0f26&acdnat=1507942672_ff44d4811825271fcf37c06314338fe1

Java Code:

https://dansesacrale.wordpress.com/2010/07/04/continued-fractions-sqrt-steps/

Continued Fractions:

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfCALC.html