Continued Fractions: Invisible Patterns

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Thomas Jackson
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Infinite Series, Periodic and Nonperiodic Decimal Expansions, Integrals

\[ \pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + - \frac{1}{10} + \frac{1}{11} + \frac{1}{12} - \frac{1}{13} \ldots \]

\[ \int_{-\infty}^{\infty} \sqrt{1 - x^2} \, dx = \frac{\pi}{2} \]

\[ \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots \rightleftharpoons \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} \]

3.1415926535897932384626433832795……..
\[ \varphi = \frac{1 + \sqrt{5}}{2} = 1.6180339887 \ldots \]

\[ \varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}} \]

\[ e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1\cdot2} + \frac{1}{1\cdot2\cdot3} + \frac{1}{1\cdot2\cdot3\cdot4} + \ldots \]

\[ e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2.718281828459\ldots \]

\[ \sqrt{2} = 1.41421356237\ldots \]
\[
\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \frac{11^2}{2 + \ldots}}}}}}
\]
EXAMPLES OF CF'S

\[ e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{6 + \ldots}}}}}}}}}} \]

\[ \Phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}}}}}} \]
\[
\frac{\pi}{2} = 1 - \frac{1}{3 - \frac{2 \cdot 3}{1 \cdot 2}} - \frac{1}{3 - \frac{4 \cdot 5}{3 \cdot 4}} - \frac{1}{3 - \frac{6 \cdot 7}{5 \cdot 6}} - \cdots
\]

\[
\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}}
\]
CONTINUED FRACTIONS TYPES

- Continued Fractions
  - Simple
  - General
    - Finite
    - Infinite
      - Periodic
      - Non-Periodic
CONTINUED FRACTIONS FORMS

General Continued Fraction

\[ a_1 + \frac{b_1}{a_2 + \frac{b_2}{a_3 + \frac{b_3}{a_4 + \frac{b_4}{a_5 + \frac{b_5}{a_6 \ldots}}} \ldots}} \]

Simple Continued Fraction (\(b_i = 1\))

\[ a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \frac{1}{a_5 + \frac{1}{a_6 \ldots}}} \ldots}} \]

= \([a_1; a_2, a_3, a_4, \ldots] = [a_1: a_2, a_3, a_4, \ldots]\]

Convergents

\[ \frac{p_i}{q_i} = [a_1; a_2, a_3, a_4, a_5, \ldots, a_n] \]
\[ \pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, ...] \]

\[ e = [2; 1, 2, 1, 1, 4, 1, 1, 6, ...] \]

\[ \varphi = [1; 1, 1, 1, 1, 1, 1, ...] \]
Consider the fraction \( \frac{47}{13} \). Can we find the continued fraction (cf.) for this finite ratio?

We should consider breaking up the fraction into mixed form.

So \( \frac{47}{13} = 3 + \frac{8}{13} \)

We need the numerator to be one for our cf. form so we can apply the following algebraic identity \( \frac{a}{b} = \frac{1}{\frac{b}{a}} \) to move forward.
\[
\frac{47}{13} = 3 + \frac{1 \ 13}{8} = 3 + \frac{1}{1 + \frac{5}{8}} = 3 + \frac{1}{1 + \frac{1}{\frac{8}{5}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{3}{5}}}
\]

\[
\frac{47}{13} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{5}{3}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{2}{3}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{2}{3}}}
\]

\[
\frac{47}{13} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{3}{2}}}} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}
\]
This procedure is an example of an algorithm called the Euclidean Algorithm which was developed by Euclid in his famous book The Elements. We can also look at the problem from geometrical and coding perspectives.
Let’s consider the calculations already performed for the fraction \[ \frac{47}{13} \].

\[
47 = 3(13) + 8 \\
13 = 1(8) + 5 \\
8 = 1(5) + 3 \\
5 = 1(3) + 2 \\
3 = 1(2) + 1 \\
2 = 2(1) + 0
\]
We continue this operation on the two numbers and it will always stop when the length of the shortest rectangle is one and the remainder after the last triangle has been divided is zero.

We can find the cf. form from the leading numbers from the list:

\[
\frac{47}{13} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}}.
\]
We all know that $\phi$ is derived from the roots of the quadratic equation

$$\phi^2 - \phi - 1 = 0$$

$$\phi = 1 + \frac{1}{\phi}$$

$$\phi = 1 + \frac{1}{1 + \frac{1}{\phi}}$$

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\phi}}}$$

$$\phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ldots}}}$$

$$\phi = [1; 1, 1, 1, 1, 1, 1, 1, \ldots]$$
We can look at the partial cfs for the fraction $\frac{47}{13}$ to find better approximations to the exact value of this fraction. These values are called convergents.

\[ 3, 3 + \frac{1}{1}, 3 + \frac{1}{1+\frac{1}{1}}, 3 + \frac{1}{1+\frac{1}{1+\frac{1}{1}}}, 3 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} \]

\[ 3, 4, \frac{7}{2}, \frac{11}{3}, \frac{18}{5}, \frac{47}{13} = 3, 4, 3.5, 3.666.., 3.6, 3.65… \]
We can generalize this procedure. Let \( c_i = \frac{p_i}{q_i} \) represent the convergents for a given fraction with \( i \geq 0 \).

So
\[
\begin{align*}
    c_1 &= \frac{p_1}{q_1} = \frac{a_1}{1} \\
    c_2 &= \frac{p_2}{q_2} = a_1 + \frac{1}{a_2} = \frac{a_1a_2 + 1}{a_2} \\
    c_3 &= \frac{p_3}{q_3} = a_1 + \frac{1}{a_2 + \frac{1}{a_3}} = \frac{a_1a_2a_3 + a_1 + a_3}{a_2a_3 + 1}
\end{align*}
\]
\[ c_4 = \frac{p_4}{q_4} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}} = \frac{a_1 a_2 a_3 a_4 + a_1 a_2 + a_1 a_4 + a_3 a_4 + 1}{a_2 a_3 a_4 + a_2 + a_4} \]

So we can “see” the following recursions for the convergent terms if we look carefully:

\[ p_i = a_i p_{i-1} + p_{i-2} \]
\[ q_i = a_i q_{i-1} + q_{i-2} \]
Using these recursion formulas, the convergents for the golden ratio $\phi$ are

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 5 & \quad 8 & \quad 13 \\
1' & \quad 1' & \quad 2' & \quad 3' & \quad 5' & \quad 18'
\end{align*}
\]

We can see that the numbers in the numerators and denominators are terms from the Fibonacci sequence!

We should notice that

\[
p_i = a_i p_{i-1} + p_{i-2} = (1)p_{i-1} + p_{i-2} = p_{i-1} + p_{i-2}
\]

\[
q_i = a_i q_{i-1} + q_{i-2} = (1)q_{i-1} + q_{i-2} = q_{i-1} + q_{i-2}
\]

and $c_i = \frac{t_{i+1}}{t_i}$ where $t_i$ are the terms of the Fibonacci sequence.
\[ \sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ldots}}}} = [1; 2, 2, 2, 2, 2, 2, \ldots] \]
Let's see how we can derive this cf representation.

\[
\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{\sqrt{2} - 1} = 1 + \frac{1}{\sqrt{2} + 1}
\]

But \(\sqrt{2} + 1 = 2 + (\sqrt{2} - 1) = 2 + \frac{1}{\sqrt{2} - 1} = 2 + \frac{1}{\sqrt{2} + 1}\)

Thus \(\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{\sqrt{2} + 1}}\)

If we use this rationalization method again repeatedly, we find the cf. of \(\sqrt{2}\) consists of a single 1 followed entirely by 2’s.

Thus \(\sqrt{2} = [1; 2, 2, 2, 2, 2, ... ]\)
Is there a pattern in the representation?

Can we generalize numbers of the form $\sqrt{n^2 + 1}$?

Let’s give it a go!
Let’s consider a simple case and suppose

\[ A = [a; b, b, b, b, b, \ldots] \]

We can rewrite \( A \) in the form

\[ A = a + \frac{1}{[b; b, b, b, b, b, \ldots]} \]

We need to determine the value of

\[ B = [b; b, b, b, b, b, \ldots] \]

Just like for \( A \), we can rewrite

\[ B = b + \frac{1}{[b; b, b, b, b, b, \ldots]} \]
So we have $B = b + \frac{1}{B}$ which can be rewritten as $B^2 - bB - 1 = 0$

Using the Quadratic Formula, we have $B = \frac{b + \sqrt{b^2 + 4}}{2}$

Thus $A = a + \frac{2}{b + \sqrt{b^2 + 4}} = a - \frac{b - \sqrt{b^2 + 4}}{2} = \frac{2a - b}{2} + \frac{\sqrt{b^2 + 4}}{2}$

So we have $\frac{2a - b}{2} + \frac{\sqrt{b^2 + 4}}{2} = [a; b, b, b, b, ...]

If $b = 2a$, then $\sqrt{a^2 + 1} = [a; 2a, 2a, 2a, 2a, ...]$
Let’s try to derive the continued fraction for \( \sqrt{3} \) on the board together!
\[ \sqrt{3} = 1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{2 + \cdots}}} \}
\]

\[ = \left[ 1; 1, 2, 1, 2, 1, 2, \ldots \right] \]

\[ . \]
## PERIODIC CF’S

<table>
<thead>
<tr>
<th>$D$</th>
<th>Continued fraction of $\sqrt{D}$</th>
<th>Period</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots]$</td>
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</tr>
<tr>
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<td>$[1, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots]$</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>$[2, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, \ldots]$</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>$[2, 1, 1, 1, 2, 1, 1, 2, 1, 1, 1, 1, 2, 1, 1, \ldots]$</td>
<td>4</td>
</tr>
<tr>
<td>11</td>
<td>$[3, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, 3, 6, \ldots]$</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>$[3, 1, 1, 1, 1, 6, 1, 1, 1, 6, 1, 1, 1, 6, 1, 1, \ldots]$</td>
<td>5</td>
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<tr>
<td>17</td>
<td>$[4, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, \ldots]$</td>
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<td>23</td>
<td>$[4, 1, 3, 1, 8, 1, 3, 1, 8, 1, 3, 1, 8, 1, 3, 1, 8, 1, \ldots]$</td>
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<td>29</td>
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<td>5</td>
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<tr>
<td>31</td>
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<td>37</td>
<td>$[6, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, 12, \ldots]$</td>
<td>1</td>
</tr>
</tbody>
</table>
What about cube roots?

These continued fractions are not periodic.

We will find their cf. representation from their decimal first.

From the decimal part, we will repeatedly invert the fractional part.
\( \sqrt[3]{2} = 1.259921 \ldots \)

\[= 1 + \frac{1}{3 + \frac{1}{1.180189 \ldots}} \]

\[= 1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{5.549736 \ldots}}} \]

\[= 1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{1.819053 \ldots}}}}} \]

\[= 1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{1 + \frac{1}{1.220922 \ldots}}}}} \]

\[= 1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{5 + \frac{1}{1 + \frac{1}{4.526491 \ldots}}}}} \]
\[ \pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots], \]
\[ 3\sqrt{2} = [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, \ldots], \]
\[ \sqrt{2} = [1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots], \]
\[ e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, \ldots]. \]
\[ \pi^{1/4} \]

Continued Fractions - Professor John Barrow

\[ \pi = \left(\frac{2143}{22}\right)^{1/4} \] is good to 3 parts in \(10^4\)!

Ramanujan knew that \(\pi^4 = [97; 2, 2, 3, 1, 16539, 1, \ldots]\)

Note that the 431st digit of \(\pi\) is 20776

\[ x + 1 = \sqrt{1 + \frac{1}{x + 1}} \]
\[ \pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + 0.63459101439547237857177\ldots}}} } \]
We can find the cube roots using the bisection method.

Easy for high school students to understand.

Can be easily coded in a programming language.

We have some simple Java programs to demonstrate.
Rational Numbers

Square Root

Cube Root

$\pi$
How do we evaluate Pi using a formula?

We use an identity that came up recently in our Math club meeting.

\[ \pi = 16 \arctan \left( \frac{1}{5} \right) - 4 \arctan \left( \frac{1}{239} \right) \]
Ramanujan proved two connections between $\pi$, $e$ and $\phi$:

$$
(\sqrt{2} + \phi - \phi) e^{2\pi/5} = 1 + \frac{e^{-2\pi}}{1 + \frac{e^{-4\pi}}{1 + \frac{e^{-6\pi}}{1 + \ldots}}}
$$

$$
(\sqrt{2} - \phi - \phi) e^{\pi/5} = 1 - \frac{e^{-\pi}}{1 + \frac{e^{-2\pi}}{1 - \frac{e^{-3\pi}}{1 + \ldots}}}
$$
A CONTINUED FRACTION APPROXIMATION OF THE GAMMA FUNCTION

\[
\Gamma(x + 1) \approx \sqrt{2\pi x} \cdot e^{-x} \left( x + \frac{1}{12x - \frac{1}{x + \frac{1}{x + \frac{1}{x + \ddots}}}} \right) \]

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REFERENCES

You Tube video: **Continued Fractions - Professor John Barrow**

https://www.youtube.com/watch?v=zCFF1l7NzVQ&t=670s

The Topsy-Turvy World of Continued Fractions:

https://www.math.brown.edu/~jhs/frintonlinechapters.pdf

Cube Root of a Number:

http://www.geeksforgeeks.org/find-cubic-root-of-a-number/

Gamma Function:


A Continued Fraction Approximation of the Gamma Function:


https://ac.elscdn.com/S0022247X12009274/1-s2.0-S0022247X12009274-main.pdf?_tid=4981f0e4-b07a-11e7-bb6a-00000aab0f26&acdnat=1507942672_ff44d4811825271fcf37e06314338fc1

Java Code:

https://dansesacrale.wordpress.com/2010/07/04/continued-fractions-sqrt-steps/

Continued Fractions:

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/efCALC.html