The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
www.cemc.uwaterloo.ca

Galois Contest

(Grade 10)

Thursday, April 12, 2012
(in North America and South America)

Friday, April 13, 2012
(outside of North America and South America)

Do not open this booklet until instructed to do so.

Time: 75 minutes | **Number of questions:** 4
Calculators are permitted | **Each question is worth 10 marks**

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided

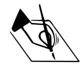
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

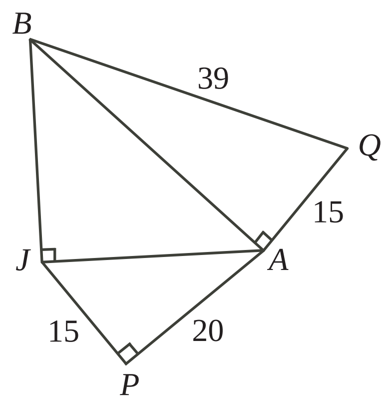
- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published in the FGH Results on our Web site, http://www.cemc.uwaterloo.ca.

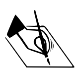
TIPS:	
1.	Please read the instructions on the front cover of this booklet.
2.	Write all answers in the answer booklet provided.
3.	For questions marked , place your answer in the appropriate box in the answer booklet and **show your work**.
4.	For questions marked , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5.	Diagrams are *not* drawn to scale. They are intended as aids only.

1. Adam and Budan are playing a game of Bocce. Each wants their ball to land closest to the jack ball. The positions of Adam's ball, A, Budan's ball, B, and the jack ball, J, are shown in the diagram.



 (a) What is the distance from A to J?

 (b) What is the distance from B to A?

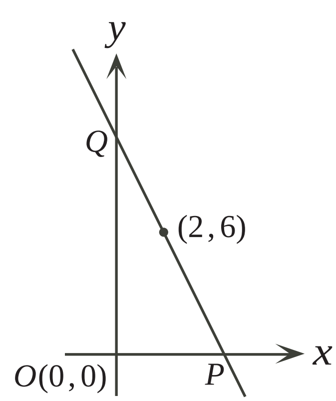
 (c) Determine whose ball is closer to the jack ball, Adam's or Budan's.

2.  (a) When the numbers $25, 5$ and 29 are taken in pairs and averaged, what are the three averages?

 (b) When the numbers $2, 6$ and n are taken in pairs and averaged, the averages are 11, 4 and 13. Determine the value of n.

 (c) There are three numbers a, b and 2. Each number is added to the average of the other two numbers. The results are $14, 17$ and 21. If $2 < a < b$, determine the values of a and b.

3. The diagram shows one of the infinitely many lines that pass through the point $(2, 6)$.

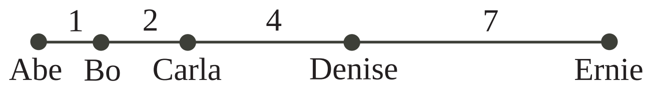


 (a) A line through the point $(2, 6)$ has slope -3. Determine the x- and y-intercepts of this line.

 (b) Another line through the point $(2, 6)$ has slope m. Determine the x- and y-intercepts of this line in terms of m.

 (c) A line through the point $(2, 6)$ has slope m, and crosses the positive x-axis at P and the positive y-axis at Q, as shown. Determine the two values of m for which $\triangle POQ$ has an area of 25.

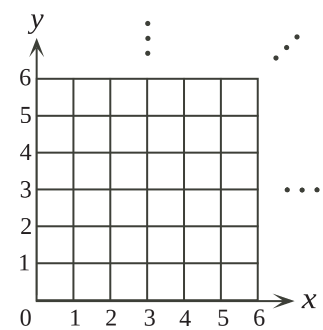
4. (a) In Town A, five students are standing at different intersections on the same east-west street, as shown. The distances between adjacent intersections are given in kilometres.



The students agree to meet somewhere on the street such that the total distance travelled by all five students is as small as possible. Where should the students meet?

(b) In Town B, there is an even number of students. The students are standing at different intersections on a straight north-south street. The students agree to meet somewhere on the street that will make the total distance travelled as small as possible. With justification, determine all possible locations where the students could meet.

(c) In Town C, the streets run north-south and east-west forming a positive xy-plane with intersections every 1 km apart, as shown. One hundred students are standing at different intersections. The first 50 students, numbered 1 to 50, stand so that the student numbered k stands at intersection $(2^k, k)$. (For example, student 5 stands at $(32, 5)$.) The remaining students, numbered 51 to 100, stand so that the student numbered j stands at intersection $(j - 50, 2j - 100)$. The students can only travel along the streets, and they agree to meet at an intersection that will make the total distance travelled by all students as small as possible. With justification, determine all possible intersections at which the students could meet.



2012 Galois Contest (English)

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

For students...

Thank you for writing the 2012 Galois Contest!
In 2011, more than 13 000 students from around the world registered to write the Fryer, Galois and Hypatia Contests.

Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2012.

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2011 Galois Contest (Grade 10)

Wednesday, April 13, 2011

1. Jackson gave the following rule to create sequences:
   "If x is a term in your sequence, then the next term in your sequence is $\dfrac{1}{1-x}$."
   For example, Mary starts her sequence with the number 3.
   The second term of her sequence is $\dfrac{1}{1-3} = \dfrac{1}{-2} = -\dfrac{1}{2}$. Her sequence is now $3, -\frac{1}{2}$.
   The third term of her sequence is $\dfrac{1}{1-(-\frac{1}{2})} = \dfrac{1}{\frac{3}{2}} = \dfrac{2}{3}$. Her sequence is now $3, -\frac{1}{2}, \frac{2}{3}$.
   The fourth term of her sequence is $\dfrac{1}{1-\frac{2}{3}} = \dfrac{1}{\frac{1}{3}} = 3$. Her sequence is now $3, -\frac{1}{2}, \frac{2}{3}, 3$.

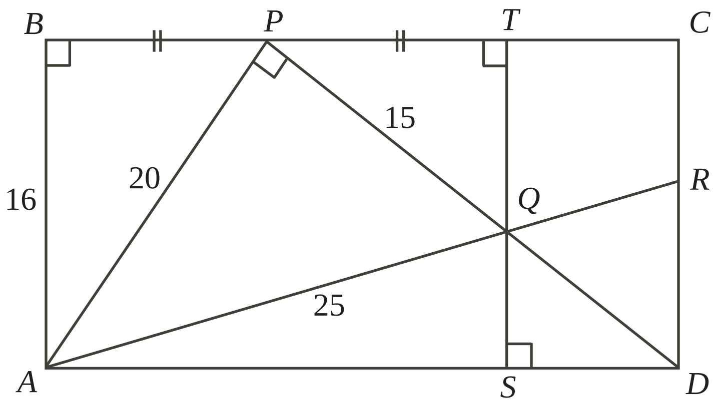
   Fabien starts his sequence with the number 2, and continues using Jackson's rule until the sequence has 2011 terms.

   (a) What is the second term of his sequence?
   (b) What is the fifth term of his sequence?
   (c) How many of the 2011 terms in Fabien's sequence are equal to 2? Explain.
   (d) Determine the sum of all of the terms in his sequence.

2. Alia has a bucket of coins. Each coin has a zero on one side and an integer greater than 0 on the other side. She randomly draws three coins, tosses them and calculates a score by adding the three numbers that appear.

   (a) On Monday, Alia draws coins with a 7, a 5 and a 10. When she tosses them, they show $7, 0$ and 10 for a score of 17. What other scores could she obtain by tossing these same three coins?
   (b) On Tuesday, Alia draws three coins and tosses them three times, obtaining scores of $60, 110$ and 130. On each of these tosses, exactly one of the coins shows a 0. Determine the maximum possible score that can be obtained by tossing these three coins.
   (c) On Wednesday, Alia draws one coin with a 25, one with a 50, and a third coin. She tosses these three coins and obtains a score of 170. Determine all possible numbers, other than zero, that could be on the third coin.

3. In rectangle $ABCD$, P is a point on BC so that $\angle APD = 90^\circ$. TS is perpendicular to BC with $BP = PT$, as shown. PD intersects TS at Q. Point R is on CD such that RA passes through Q. In $\triangle PQA$, $PA = 20$, $AQ = 25$ and $QP = 15$.



(a) Determine the lengths of BP and QT.

(b) Show that $\triangle PQT$ and $\triangle DQS$ are similar. That is, show that the corresponding angles in these two triangles are equal.

(c) Determine the lengths of QS and SD.

(d) Show that $QR = RD$.

4. For a positive integer n, the n^{th} triangular number is $T(n) = \dfrac{n(n+1)}{2}$.

For example, $T(3) = \frac{3(3+1)}{2} = \frac{3(4)}{2} = 6$, so the third triangular number is 6.

(a) There is one positive integer a so that $T(4) + T(a) = T(10)$. Determine a.

(b) Determine the smallest integer $b > 2011$ such that $T(b+1) - T(b) = T(x)$ for some positive integer x.

(c) If $T(c) + T(d) = T(e)$ and $c + d + e = T(28)$, then show that $cd = 407(c + d - 203)$.

(d) Determine all triples (c, d, e) of positive integers such that $T(c) + T(d) = T(e)$ and $c + d + e = T(28)$, where $c \leq d \leq e$.

2010 Galois Contest (Grade 10)

Friday, April 9, 2010

1. Emily's old showerhead used 18 L of water per minute.
   She installs a new showerhead that uses 13 L per minute.

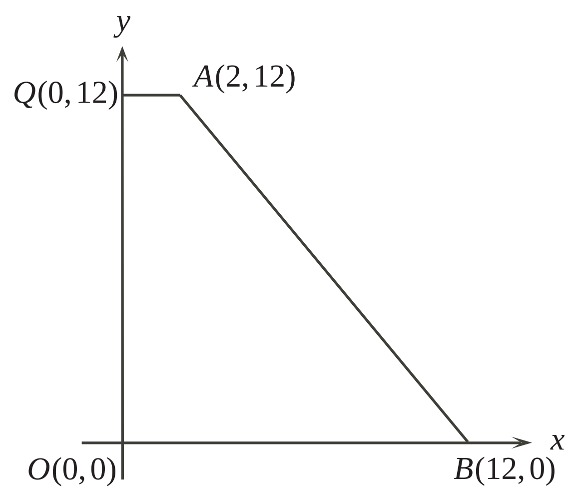
   (a) When Emily takes a bath, she uses 260 L of water. Using the new showerhead, what length of shower, in minutes, uses 260 L of water?

   (b) How much *less* water is used for a 10 minute shower with the new showerhead than with the old showerhead?

   (c) Emily is charged 8 cents per 100 L of water that she uses. Using the new showerhead instead of the old showerhead saves water and so saves Emily money. How much money does Emily *save* in water costs for a 15 minute shower?

   (d) How many minutes of showering, using the new showerhead, will it take for Emily to have saved $30 in water costs?

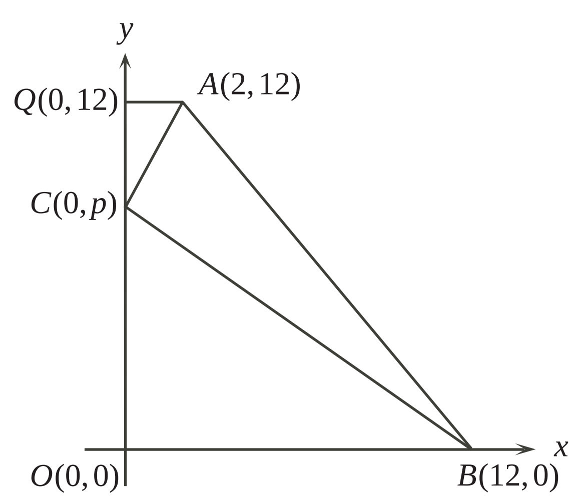
2. (a) Quadrilateral $QABO$ is constructed as shown. Determine the area of $QABO$.



   (b) Point $C(0, p)$ lies on the y-axis between $Q(0, 12)$ and $O(0, 0)$ as shown. Determine an expression for the area of $\triangle COB$ in terms of p.

   (c) Determine an expression for the area of $\triangle QCA$ in terms of p.

   (d) If the area of $\triangle ABC$ is 27, determine the value of p.



3. (a) Solve the system of equations algebraically for (x, y):

$$\begin{aligned} x + y &= 42 \\ x - y &= 10 \end{aligned}$$

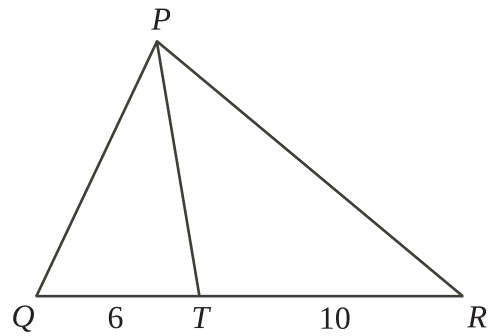
(b) Suppose that p is an even integer and that q is an odd integer. Explain why the system of equations

$$\begin{aligned} x + y &= p \\ x - y &= q \end{aligned}$$

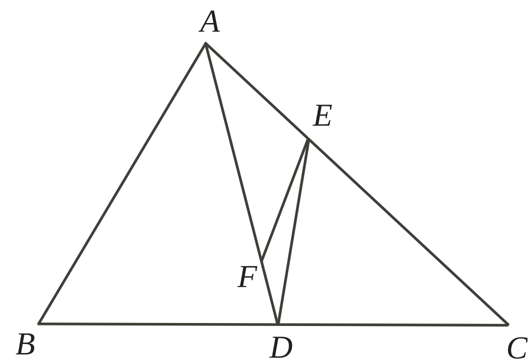
has no positive integer solutions (x, y).

(c) Determine all pairs of positive integers (x, y) that satisfy the equation $x^2 - y^2 = 420$.

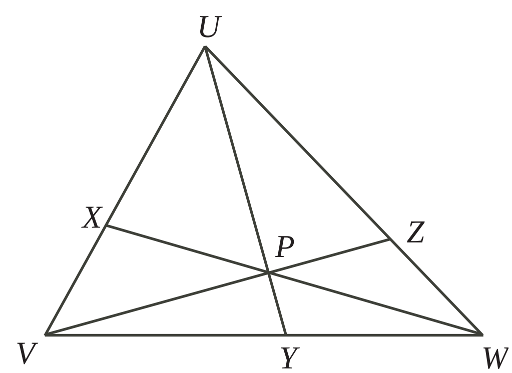
4. (a) In $\triangle PQR$, point T is on side QR such that $QT = 6$ and $TR = 10$. Explain why the ratio of the area of $\triangle PQT$ to the area of $\triangle PTR$ is $3 : 5$.



(b) In $\triangle ABC$, point D is the midpoint of side BC. Point E is on AC such that $AE : EC = 1 : 2$. Point F is on AD such that $AF : FD = 3 : 1$. If the area of $\triangle DEF$ is 17, determine the area of $\triangle ABC$.



(c) In the diagram, points X, Y and Z are on the sides of $\triangle UVW$, as shown. Line segments UY, VZ and WX intersect at P. Point Y is on VW such that $VY : YW = 4 : 3$. If $\triangle PYW$ has an area of 30 and $\triangle PZW$ has an area of 35, determine the area of $\triangle UXP$.



2009 Galois Contest (Grade 10)

Wednesday, April 8, 2009

1. Alex counts the number of students in her class with each hair colour, and summarizes the results in the following table:

Hair Colour	Number of Students
Blonde	8
Brown	7
Red	3
Black	2

   (a) What percentage of students in the class have blonde hair?

   (b) What percentage of students in the class have red or black hair?

   (c) How many students in the class with blonde hair would have to dye their hair black for the percentage of students in the class with black hair to be 20%?

   (d) How many students with red hair would have to join the class so the percentage of students in the class with red hair is equal to 32%?

2. A square has vertices with coordinates $A(6,9)$, $B(12,6)$, $C(t,0)$, and $D(3,3)$.

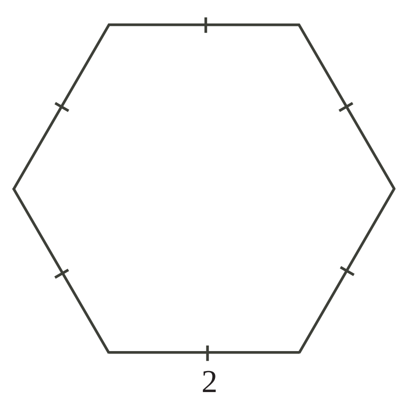
   (a) Determine the value of t, the x-coordinate of vertex C.

   (b) A line is drawn through $O(0,0)$ and D. This line meets AB at E. Determine the coordinates of E.

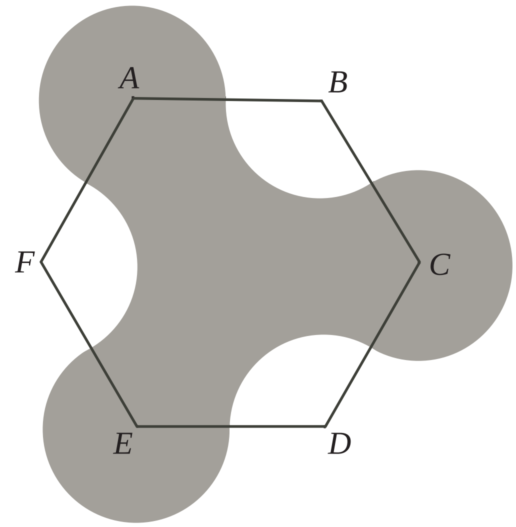
   (c) Determine the perimeter of quadrilateral $EBCD$.

3. (a) Find the area of an equilateral triangle with side length 2.

   (b) Determine the area of a regular hexagon with side length 2.



(c) In the diagram, regular hexagon $ABCDEF$ has sides of length 2. Using A, C and E as centres, portions of circles with radius 1 are drawn outside the hexagon. Using B, D and F as centres, portions of circles with radius 1 are drawn inside the hexagon. These six circular arcs join together to form a curve. Determine the area of the shaded region enclosed by this curve.



4. If m is a positive integer, the symbol $m!$ is used to represent the product of the integers from 1 to m. That is, $m! = m(m-1)(m-2)\cdots(3)(2)(1)$. For example, $5! = 5(4)(3)(2)(1)$ or $5! = 120$.
Some positive integers n can be written in the form

$$n = a(1!) + b(2!) + c(3!) + d(4!) + e(5!).$$

In addition, each of the following conditions is satisfied:

- a, b, c, d, and e are integers
- $0 \leq a \leq 1$
- $0 \leq b \leq 2$
- $0 \leq c \leq 3$
- $0 \leq d \leq 4$
- $0 \leq e \leq 5$.

(a) Determine the largest positive integer N that can be written in this form.

(b) Write $n = 653$ in this form.

(c) Prove that all integers n, where $0 \leq n \leq N$, can be written in this form.

(d) Determine the sum of all integers n that can be written in this form with $c = 0$.

2008 Galois Contest (Grade 10)

Wednesday, April 16, 2008

1. Three positive integers a, b and x form an O'Hara triple (a, b, x) if $\sqrt{a} + \sqrt{b} = x$. For example, $(1, 4, 3)$ is an O'Hara triple because $\sqrt{1} + \sqrt{4} = 3$.

   (a) If $(36, 25, x)$ is an O'Hara triple, determine the value of x.

   (b) If $(a, 9, 5)$ is an O'Hara triple, determine the value of a.

   (c) Determine the five O'Hara triples with $x = 6$. Explain how you found these triples.

2. (a) Determine the equation of the line passing through the points $P(0, 5)$ and $Q(6, 9)$.

   (b) A line, through Q, is perpendicular to PQ. Determine the equation of the line.

   (c) The line from (b) crosses the x-axis at R. Determine the coordinates of R.

   (d) Determine the area of right-angled $\triangle PQR$.

3. (a) A class of 20 students was given a two question quiz. The results are listed below:

Question number	Number of students who answered correctly
1	18
2	14

   Determine the smallest possible number and the largest possible number of students that could have answered both questions correctly. Explain why these are the smallest and largest possible numbers.

   (b) A class of 20 students was given a three question quiz. The results are listed below:

Question number	Number of students who answered correctly
1	18
2	14
3	12

   Determine the smallest possible number and the largest possible number of students that could have answered all three questions correctly. Explain why these are the smallest and largest possible numbers.

   (c) A class of 20 students was given a three question quiz. The results are listed below:

Question number	Number of students who answered correctly
1	x
2	y
3	z

   where $x \geq y \geq z$ and $x + y + z \geq 40$.
   Determine the smallest possible number of students who could have answered all three questions correctly in terms of x, y and z.

4. Carolyn and Paul are playing a game starting with a list of the integers 1 to n. The rules of the game are:

- Carolyn always has the first turn.
- Carolyn and Paul alternate turns.
- On each of her turns, Carolyn must remove one number from the list such that this number has at least one positive divisor other than itself remaining in the list.
- On each of his turns, Paul must remove from the list all of the positive divisors of the number that Carolyn has just removed.
- If Carolyn cannot remove any more numbers, then Paul removes the rest of the numbers.

For example, if $n = 6$, a possible sequence of moves is shown in this chart:

Player	Number(s) removed	Number(s) remaining	Notes
Carolyn	4	1, 2, 3, 5, 6	
Paul	1, 2	3, 5, 6	
Carolyn	6	3, 5	She could not remove 3 or 5
Paul	3	5	
Carolyn	None	5	Carolyn cannot remove any number
Paul	5	None	

In this example, the sum of the numbers removed by Carolyn is $4 + 6 = 10$ and the sum of the numbers removed by Paul is $1 + 2 + 3 + 5 = 11$.

(a) Suppose that $n = 6$ and Carolyn removes the integer 2 on her first turn. Determine the sum of the numbers that Carolyn removes and the sum of the numbers that Paul removes.

(b) If $n = 10$, determine Carolyn's maximum possible final sum. Prove that this sum is her maximum possible sum.

(c) If $n = 14$, prove that Carolyn cannot remove 7 numbers.

2007 Galois Contest (Grade 10)

Wednesday, April 18, 2007

1. Jim shops at a strange fruit store. Instead of putting prices on each item, the mathematical store owner will answer questions about combinations of items.

   (a) In Aisle 1, Jim receives the following answers to his questions:

Jim's Question	Answer
What is the sum of the prices of an Apple and a Cherry?	62 cents
What is the sum of the prices of a Banana and a Cherry?	66 cents

   What is difference between the prices of an Apple and a Banana? Which has a higher price? Explain how you obtained your answer.

   (b) In Aisle 2, Jim receives the following answers to his questions:

Jim's Question	Answer
What is the sum of the prices of a Mango and a Nectarine?	60 cents
What is the sum of the prices of a Pear and a Nectarine?	60 cents
What is the sum of the prices of a Mango and a Pear?	68 cents

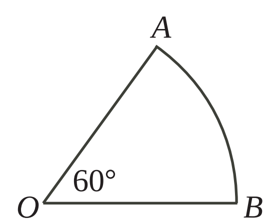
   What is the price of a Pear? Explain how you obtained your answer.

   (c) In Aisle 3, Jim receives the following answers to his questions:

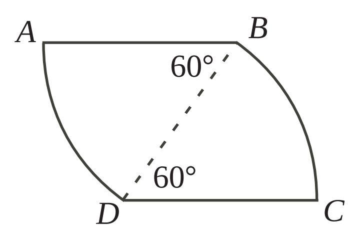
Jim's Question	Answer
What is the sum of the prices of a Tangerine and a Lemon?	60 cents
How much more does a Tangerine cost than a Grapefruit?	6 cents
What is the sum of the prices of Grapefruit, a Tangerine and a Lemon?	94 cents

   What is the price of a Lemon? Explain how you obtained your answer.

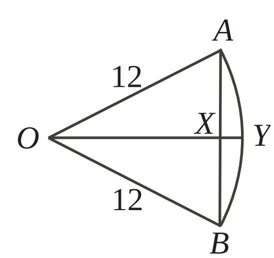
2. (a) In the diagram, what is the perimeter of the sector of the circle with radius 12? Explain how you obtained your answer.



   (b) Two sectors of a circle of radius 12 are placed side by side, as shown. Determine the *area* of figure $ABCD$. Explain how you obtained your answer.

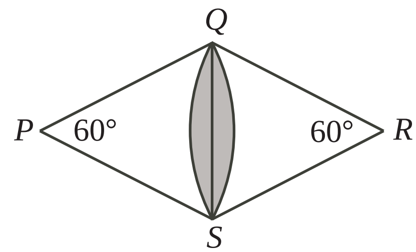


   (c) In the diagram, AOB is a sector of a circle with $\angle AOB = 60°$. OY is drawn perpendicular to AB and intersects AB at X. What is the length of XY? Explain how you obtained your answer.

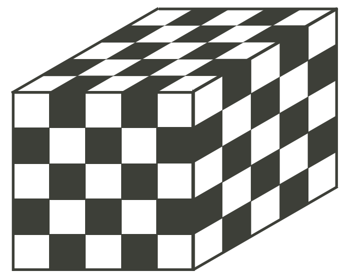


   (d) See over...

(d) Two sectors of a circle of radius 12 overlap as shown. Determine the area of the shaded region. Explain how you obtained your answer.



3. (a) Each face of a 5 by 5 by 5 wooden cube is divided into 1 by 1 squares. Each square is painted black or white, as shown. Next, the cube is cut into 1 by 1 by 1 cubes. How many of these cubes have *at least* two painted faces? Explain how you obtained your answer.



(b) A $(2k + 1)$ by $(2k + 1)$ by $(2k + 1)$ cube, where k is a positive integer, is painted in the same manner as the 5 by 5 by 5 cube with white squares in the corners. Again, the cube is cut into 1 by 1 by 1 cubes.

i. In terms of k, how many of these cubes have *exactly* two white faces? Explain how you obtained your answer.

ii. Prove that there is no value of k for which the number of cubes having *at least* two white faces is 2006.

4. Jill has a container of small cylindrical rods in six different colours. Each colour of rod has a different length as summarized in the chart.

Colour	Length
Green	3 cm
Pink	4 cm
Yellow	5 cm
Black	7 cm
Violet	8 cm
Red	9 cm

These rods can be attached together to form a pole.
There are 2 ways to choose a set of yellow and green rods that will form a pole 29 cm in length: 8 green rods and 1 yellow rod OR 3 green rods and 4 yellow rods.

(a) How many different sets of yellow and green rods can be chosen that will form a pole 62 cm long? Explain how you obtained your answer.

(b) Among the green, yellow, black and red rods, find, with justification, two colours for which it is impossible to make a pole 62 cm in length using only rods of those two colours.

(c) If at least 81 rods of each of the colours green, pink, violet, and red must be used, how many different sets of rods of these four colours can be chosen that will form a pole 2007 cm in length? Explain how you got your answer.

2006 Galois Contest (Grade 10)
Thursday, April 20, 2006

1. A hat contains six slips of paper numbered from 1 to 6. Amelie and Bob each choose three slips from the hat without replacing any of the slips. Each of them adds up the numbers on his slips.

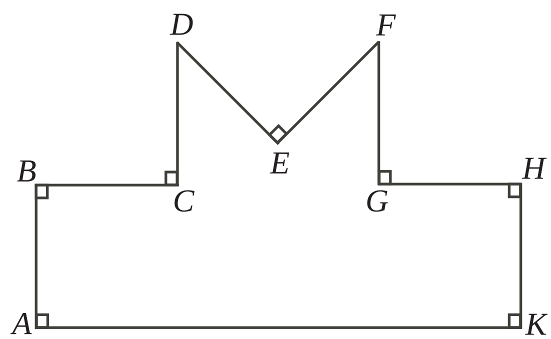
   (a) Determine the largest possible difference between Amelie's total and Bob's total. Explain how you found this difference.

   (b) List all possible groups of three slips that Amelie can choose so that her total is one more than Bob's total.

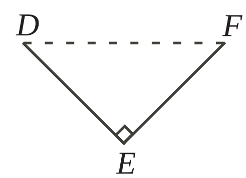
   (c) Explain why it is impossible for Amelie and Bob to have the same total no matter which three slips each chooses.

   (d) If more slips of paper are added to the hat, numbered consecutively from 7 to n, what is the smallest value of $n > 6$ so that Amelie and Bob can each choose half of the slips numbered from 1 to n and obtain the same total? Explain why this value of n works.

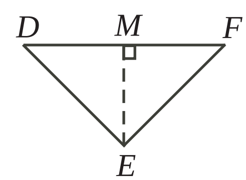
2. In the diagram, AB, BC, CD, DE, EF, FG, GH, and HK all have length 4, and all angles are right angles, with the exception of the angles at D and F.



   (a) Determine the length of DF.



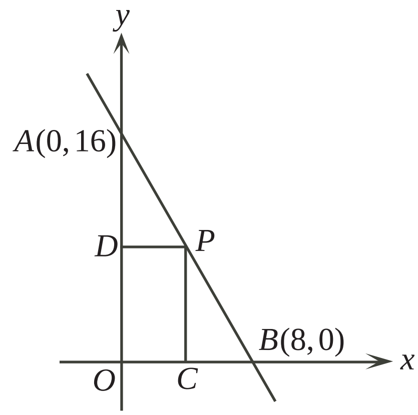
   (b) If perpendicular EM is drawn from E to DF, what is the length of EM? Explain how you got your answer.



   (c) If perpendicular EP is drawn from E to AK, what is the length of EP? Explain how you got your answer.

   (d) What is the area of figure $ABCDEFGHK$? Explain how you got your answer.

3. In the diagram, a line is drawn through the points $A(0,16)$ and $B(8,0)$. Point P is chosen in the first quadrant on the line through A and B. Points C and D are then chosen on the x-axis and y-axis, respectively, so that $PDOC$ is a rectangle.



   (a) Determine the equation of the line through A and B.

   (b) Determine the coordinates of the point P so that $PDOC$ is a square.

   (c) Determine the coordinates of all points P that can be chosen so that the area of rectangle $PDOC$ is 30.

4. (a) When the number 14 has its digits reversed to form the number 41, it is increased by 27. Determine all 2-digit numbers which are increased by 27 when their digits are reversed.

   (b) Choose any three-digit integer $\underline{a}\,\underline{b}\,\underline{c}$ whose digits are all different.
   (When a three-digit integer is written in terms of its digits as $\underline{a}\,\underline{b}\,\underline{c}$, it means the integer is $100a + 10b + c$.)
   Reverse the order of the digits to get a new three-digit integer $\underline{c}\,\underline{b}\,\underline{a}$.
   Subtract the smaller of these integers from the larger to obtain a three-digit integer $\underline{r}\,\underline{s}\,\underline{t}$, where r is allowed to be 0.
   Reverse the order of the digits of this integer to get the integer $\underline{t}\,\underline{s}\,\underline{r}$.
   Prove that, no matter what three-digit integer $\underline{a}\,\underline{b}\,\underline{c}$ you start with, $\underline{r}\,\underline{s}\,\underline{t} + \underline{t}\,\underline{s}\,\underline{r} = 1089$.

   (c) Suppose that $N = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ is a four-digit integer with $a \le b \le c \le d$.
   When the order of the digits of N is reversed to form the integer M, N is increased by P.
   (Again, the first digit of P is allowed to be 0.)
   When the order of the digits of P is reversed, an integer Q is formed.
   Determine, with justification, all possible values of $P + Q$.

2005 Galois Contest (Grade 10)

Wednesday, April 20, 2005

1. An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant.
For example, the sequence 2, 11, 20, 29, ... is an arithmetic sequence.
(The "..." indicates that this sequence continues without ever ending.)

   (a) Find the 11th term in the arithmetic sequence 17, 22, 27, 32, ...

   (b) Explain why there is no number which occurs in each of the following arithmetic sequences:

   17, 22, 27, 32, ...
   13, 28, 43, 58, ...

   (c) Find a number between 400 and 420 which occurs in both of the following arithmetic sequences:

   17, 22, 27, 32, ...
   16, 22, 28, 34, ...

   Explain how you got your answer.

2. Emilia and Omar are playing a game in which they take turns placing numbered tiles on the grid shown.

Emilia starts the game with six tiles: 1, 2, 3, 4, 5, and 6.

Omar also starts the game with six tiles: 1, 2, 3, 4, 5, and 6.

Once a tile is placed, it cannot be moved.
After all of the tiles have been placed, Emilia scores one point for each row that has an even sum and one point for each column that has an even sum. Omar scores one point for each row that has an odd sum and one point for each column that has an odd sum. For example, if the game ends with the tiles placed as shown below, then Emilia will score 5 points and Omar 2 points.

3	1	2	4
5	5	2	4
1	3	6	6

   (a) In a game, after Omar has placed his second last tile, the grid appears as shown below. Starting with the partially completed game shown, give a final placement of tiles for which Omar scores more points than Emilia. (You do not have to give a strategy, simply fill in the final grid.)

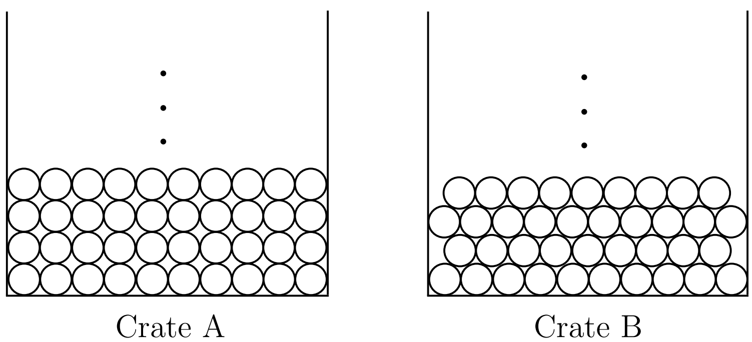
1	3		5
3	1	2	2
	4	4	

(b) Explain why it is impossible for Omar and Emilia to score the same number of points in any game.

(c) In the partially completed game shown below, it is Omar's turn to play and he has a 2 and a 5 still to place. Explain why Omar cannot score more points than Emilia, no matter where he places the 5.

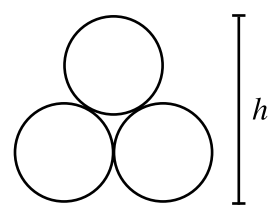
1		3	6
	5		4
3		1	6

3. Two identical rectangular crates are packed with cylindrical pipes, using different methods. Each pipe has diameter 10 cm. A side view of the first four rows of each of the two different methods of packing is shown below.



(a) If 200 pipes are packed in each of the two crates, how many rows of pipes are there in each crate? Explain your answer.

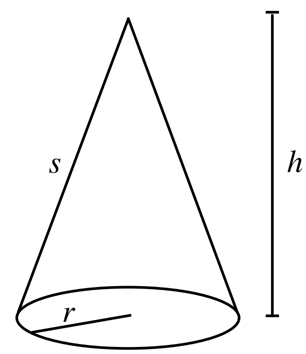
(b) Three pipes from Crate B are shown. Determine the height, h, of this pile of 3 pipes. Explain your answer.



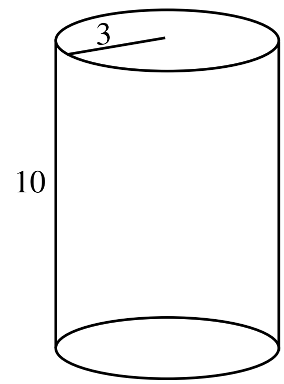
(c) After the crates have been packed with 200 pipes each, what is the difference in the total heights of the two packings? Explain your answer.

4. The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.

The *total* surface area of a cone with height h, slant height s, and radius r is $\pi r^2 + \pi r s$.



(a) A cylinder has a height of 10 and a radius of 3. Determine the *total* surface area, including the two ends, of the cylinder, and also determine the volume of the cylinder.



(b) A cone, a cylinder and a sphere all have radius r. The height of the cylinder is H and the height of the cone is h. The cylinder and the sphere have the same volume. The cone and the cylinder have the same total surface area. Prove that h and H cannot both be integers.

2004 Galois Contest (*Grade 10*)

Thursday, April 15, 2004

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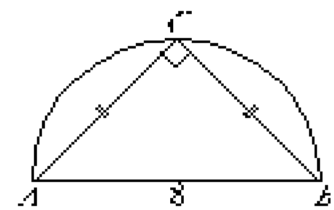
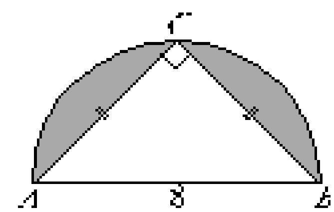
1. The Galois Group is giving out four types of prizes, valued at \$5, \$25, \$125 and \$625.
   (a) The Group gives out at least one of each type of prize. If five prizes are given out with a total value of \$905, how many of each type of prize is given out? Explain how you got your answer.
   (b) If the Group gives out at least one of each type of prize and five prizes in total, determine the other three possible total values it can give out. Explain how you got your answer.
   (c) There are two ways in which the Group could give away prizes totalling \$880 while making sure to give away at least one and at most six of each prize. Determine the two ways of doing this, and explain how you got your answer.

2. In the diagram, the semicircle has diameter $AB = 8$. Point C is on the semicircle so that triangle ABC is isosceles and right-angled.

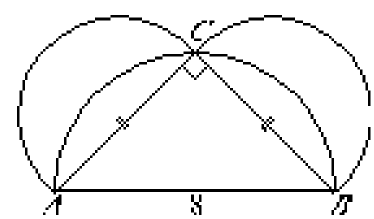


   (a) Determine the area of triangle ABC.

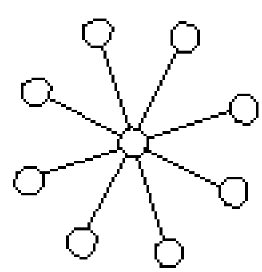
   (b) The two regions inside the semicircle but outside the triangle are shaded. Determine the total area of the two shaded regions.



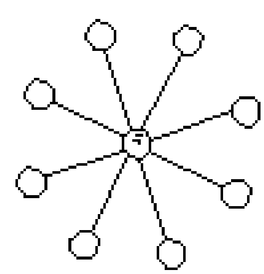
   (c) Semicircles are drawn on AC and CB, as shown. Show that:
   $(\text{Area of semicircle drawn on } AB)$
   $= (\text{Area of semicircle drawn on } AC) + (\text{Area of semicircle drawn on } CB)$



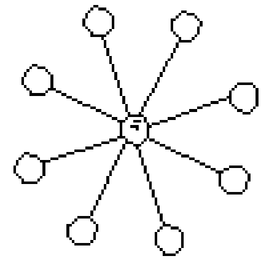
3. In "The Sun Game", two players take turns placing discs numbered 1 to 9 in the circles on the board. Each number may only be used once. The object of the game is to be the first to place a disc so that the sum of the 3 numbers along a line through the centre circle is 15.



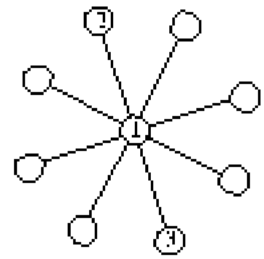
   (a) If Avril places a 5 in the centre circle and then Bob places a 3, explain how Avril can win on her next turn.



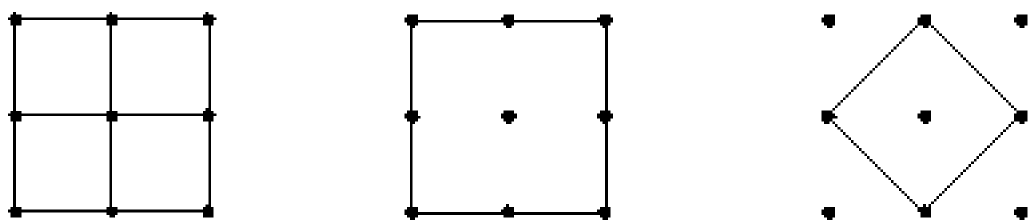
(b) If Avril starts by placing a 5 in the centre circle, show that whatever Bob does on his first turn, Avril can always win on her next turn.



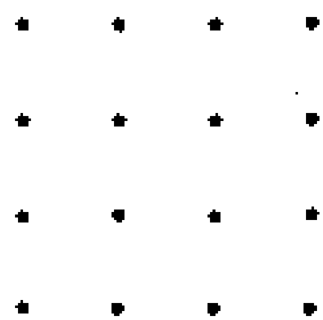
(c) If the game is in the position shown and Bob goes next, show that however Bob plays, Avril can win this game.



4. A 3 by 3 grid has dots spaced 1 unit apart both horizontally and vertically. Six squares of various side lengths can be formed with corners on the dots, as shown.



(a) Given a similar 4 by 4 grid of dots, there is a total of 20 squares of five different sizes that can be formed with corners on the dots. Draw one example of each size and indicate the number of squares there are of that size.



(b) In a 10 by 10 grid of dots, the number of squares that can be formed with side length $\sqrt{29}$ is two times the number of squares that can be formed with side length 7. Explain why this is true.

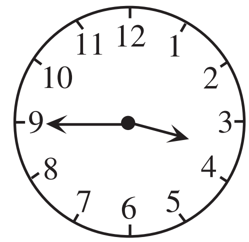
(c) Show that the total number of squares that can be formed in a 10 by 10 grid is
$1(9^2)+2(8^2)+3(7^2)+4(6^2)+5(5^2)+6(4^2)+7(3^2)+8(2^2)+9(1^2)$.

2003 Galois Contest *(Grade 10)*

Wednesday, April 16, 2003

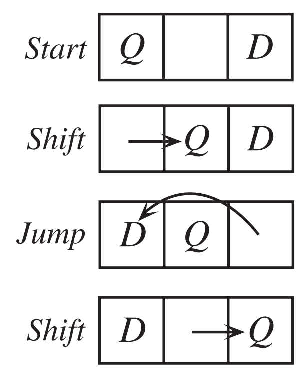
1. (a) The *sum of the squares* of 5 consecutive positive integers is 1815. What is the largest of these integers?
   (b) Show that the sum of the squares of any 5 consecutive integers is divisible by 5.

2. Professor Cuckoo mistakenly thinks that the angle between the minute hand and the hour hand of a clock at 3:45 is 180°.
   (a) Through how many degrees does the hour hand pass as the time changes from 3:00 p.m. to 3:45 p.m.?
   (b) Show that the Professor is wrong by determining the exact angle between the hands of a clock at 3:45.
   (c) At what time between 3:00 and 4:00 will the angle between the hands be 180°?



3. In the game "Switch", the goal is to make the dimes (D) and quarters (Q) switch spots. The starting position of the game with 1 quarter and 1 dime is shown below. Allowable moves are:
   (i) If there is a vacant spot beside a coin then you may *shift* to that space.
   (ii) You may *jump* a quarter with a dime or a dime with a quarter if the space on the other side is free.

   The game shown in the diagram takes three moves.

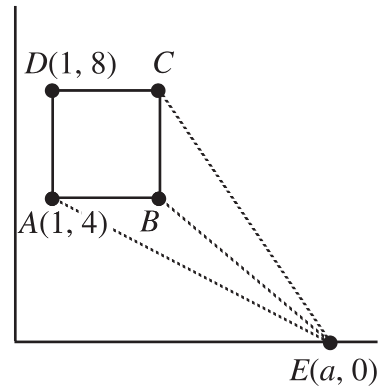


   (a) Complete the diagram to demonstrate how the game of "Switch" that starts with 2 quarters and 2 dimes can be played in 8 moves.

Q	*Q*		*D*	*D*
D	*D*		*Q*	*Q*

   (b) By considering the number of required *shifts* and *jumps*, explain why the game with 3 quarters and 3 dimes cannot be played in fewer than 15 moves.

4. In the diagram, $ABCD$ is a square and the coordinates of A and D are as shown.



(a) The point $E(a, 0)$ is on the x-axis so that the triangles CBE and ABE lie entirely outside the square $ABCD$. For what value of a is the sum of the areas of triangles CBE and ABE equal to the area of square $ABCD$?

(b) The point F is on the line passing through the points $M(6, -1)$ and $N(12, 2)$ so that the triangles CBF and ABF lie entirely outside the square $ABCD$. Determine the coordinates of the point F if the sum of the areas of triangle CBF and ABF equals the area of square $ABCD$.

Extensions (Attempt these only when you have completed as much as possible of the four main problems.)

Extension to Problem 1:
The number 1815 is also the sum of 5 consecutive positive integers. Find the next number larger than 1815 which is the sum of 5 consecutive integers and also the sum of the squares of 5 consecutive integers.

Extension to Problem 2:
The assumption might be made that there are 24 times during any 12 hour period when the angle between the hour hand and the minute hand is 90°. This is not the case. Determine the actual number of times that the angle between the hour and minute hands is 90°.

Extension to Problem 3:
Explain why the game with n quarters and n dimes cannot be played in fewer than $n(n+2)$ moves.

Extension to Problem 4:
Find the set of all points $P(x, y)$ which satisfy the conditions that the triangles CBP and ABP lie entirely outside the square $ABCD$ and the sum of the areas of triangles CBP and ABP equals the area of square $ABCD$.