The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2017 Fryer Contest

Wednesday, April 12, 2017
(in North America and South America)

Thursday, April 13, 2017
(outside of North America and South America)

Solutions

1. (a) Red pens are sold in packages of 6 pens.
Therefore, 5 packages of red pens contain $5 \times 6 = 30$ pens.
Blue pens are sold in packages of 9 pens.
Therefore, 3 packages of blue pens contain $3 \times 9 = 27$ pens.
Altogether, Igor bought $30 + 27 = 57$ pens.

(b) Robin bought 21 packages of red pens which contain $21 \times 6 = 126$ pens.
Of the 369 pens that Robin bought, the number of blue pens was $369 - 126 = 243$.
Blue pens are sold in packages of 9, so the number of packages of blue pens bought by Robin was $243 \div 9 = 27$.

(c) *Solution 1*
Let the number of packages of red pens bought by Susan be r, for some whole number r.
Let the number of packages of blue pens bought by Susan be b, for some whole number b.
Thus, Susan buys $6r$ red pens and $9b$ blue pens.
If Susan bought exactly 31 pens, then $6r + 9b = 31$.
Factoring the left side of this equation, we get $3(2r + 3b) = 31$.
Since r and b are whole numbers, then $2r + 3b$ is also a whole number, and so the left side of the equation is a multiple of 3.
Since the right side, 31, is not a multiple of 3, there is no solution to the equation $6r + 9b = 31$ for whole numbers r and b.
Therefore, it is not possible for Susan to buy exactly 31 pens.

Solution 2
Let the number of packages of red pens bought by Susan be r, for some whole number r.
Let the number of packages of blue pens bought by Susan be b, for some whole number b.
Thus, Susan buys $6r$ red pens and $9b$ blue pens.
If Susan bought exactly 31 pens, then $6r + 9b = 31$.
The smallest possible value for b is 0, and the largest possible value for b is 3 since if $b \geq 4$, then the number of pens is greater than or equal to $4 \times 9 = 36$.
Solving the equation $6r + 9b = 31$ for r, we get $6r = 31 - 9b$, and so $r = \dfrac{31 - 9b}{6}$.
In the table below, we use this equation to determine the value of r given that b can equal $0, 1, 2$, or 3.

b	$r = \dfrac{31 - 9b}{6}$
0	$r = \dfrac{31 - 9(0)}{6} = \dfrac{31}{6}$
1	$r = \dfrac{31 - 9(1)}{6} = \dfrac{22}{6}$
2	$r = \dfrac{31 - 9(2)}{6} = \dfrac{13}{6}$
3	$r = \dfrac{31 - 9(3)}{6} = \dfrac{4}{6}$

For each of the possible values for b, the resulting value of r is not a whole number.

Thus, there is no solution to the equation $6r + 9b = 31$ for whole numbers r and b, and so it is not possible for Susan to buy exactly 31 pens.

2. (a) Expressing $\frac{1}{5}$ and $\frac{1}{4}$ with a common denominator of 40, we get $\frac{1}{5} = \frac{8}{40}$ and $\frac{1}{4} = \frac{10}{40}$.
We require that $\frac{n}{40} > \frac{8}{40}$ and $\frac{n}{40} < \frac{10}{40}$, thus $n > 8$ and $n < 10$.
The only integer n that satisfies both of these inequalities is $n = 9$.

(b) Expressing $\frac{m}{8}$ and $\frac{1}{3}$ with a common denominator of 24, we require $\frac{3m}{24} > \frac{8}{24}$ and so $3m > 8$ or $m > \frac{8}{3}$.
Since $\frac{8}{3} = 2\frac{2}{3}$ and m is an integer, then $m \geq 3$.
Expressing $\frac{m+1}{8}$ and $\frac{2}{3}$ with a common denominator of 24, we require $\frac{3(m+1)}{24} < \frac{16}{24}$ or $3m + 3 < 16$ or $3m < 13$, and so $m < \frac{13}{3}$.
Since $\frac{13}{3} = 4\frac{1}{3}$ and m is an integer, then $m \leq 4$.
The integer values of m which satisfy $m \geq 3$ and $m \leq 4$ are $m = 3$ and $m = 4$.

(c) At the start of the weekend, Fiona has played 30 games and has w wins, so her win ratio is $\frac{w}{30}$.
Fiona's win ratio at the start of the weekend is greater than $0.5 = \frac{1}{2}$, and so $\frac{w}{30} > \frac{1}{2}$.
Since $\frac{1}{2} = \frac{15}{30}$, then we get $\frac{w}{30} > \frac{15}{30}$, and so $w > 15$.
During the weekend Fiona plays five games giving her a total of $30 + 5 = 35$ games played.
Since she wins three of these games, she now has $w + 3$ wins, and so her win ratio is $\frac{w+3}{35}$.
Fiona's win ratio at the end of the weekend is less than $0.7 = \frac{7}{10}$, and so $\frac{w+3}{35} < \frac{7}{10}$.
Rewriting this inequality with a common denominator of 70, we get $\frac{2(w+3)}{70} < \frac{49}{70}$ or $2(w + 3) < 49$ or $2w + 6 < 49$ or $2w < 43$, and so $w < \frac{43}{2}$.
Since $\frac{43}{2} = 21\frac{1}{2}$ and w is an integer, then $w \leq 21$.
The integer values of w which satisfy $w > 15$ and $w \leq 21$ are $w = 16, 17, 18, 19, 20, 21$.

3. (a) Chords DE and FG intersect at X, and so $(DX)(EX) = (FX)(GX)$ or $(DX)(8) = (6)(4)$.
Solving this equation for DX, we get $DX = \frac{(6)(4)}{8} = \frac{24}{8} = 3$.
The length of DX is 3.

(b) Chords JK and LM intersect at X, and so $(JX)(KX) = (LX)(MX)$ or $(8y)(10) = (16)(y + 9)$.
Solving this equation, we get $80y = 16y + 144$ or $64y = 144$, and so $y = \frac{144}{64} = \frac{9}{4}$.

(c) Chords PQ and ST intersect at U, and so $(PU)(QU) = (SU)(TU)$.
Since $TU = TV + UV$, then $TU = 6 + n$.

Substituting values into $(PU)(QU) = (SU)(TU)$, we get $(m)(5) = (3)(6 + n)$, and so $5m = 18 + 3n$.
Chords PR and ST intersect at V, and so $(PV)(RV) = (TV)(SV)$.
Since $SV = SU + UV$, then $SV = 3 + n$.
Substituting values into $(PV)(RV) = (TV)(SV)$, we get $(n)(8) = (6)(3 + n)$, and so $8n = 18 + 6n$ or $2n = 18$ or $n = 9$.
Substituting $n = 9$ into $5m = 18 + 3n$, we get $5m = 18 + 3(9)$ or $5m = 45$, and so $m = 9$.
Therefore, $m = 9$ and $n = 9$.

4. (a) Dave, Yona and Tam have 6, 4 and 8 candies, respectively.
Since they each have an even number of candies, then no candies are discarded.
During Step 2, Dave gives half of his 6 candies to Yona and accepts half of Tam's 8 candies so that he now has $6 - 3 + 4 = 7$ candies.
Similarly, Yona gives half of her 4 candies to Tam and accepts half of Dave's 6 candies so that she now has $4 - 2 + 3 = 5$ candies.
Tam gives half of his 8 candies to Dave and accepts half of Yona's 4 candies so that he now has $8 - 4 + 2 = 6$ candies.
Since Dave has 7 candies, and Yona has 5 candies, they each discard one candy while Tam, who has an even number of candies, does nothing.
These next two steps are summarized in the table to the right.

Following the given procedure, we continue the table until the procedure ends, as shown.
When the procedure ends, Dave, Yona and Tam each have 4 candies.

	Dave	Yona	Tam
Start	3	7	10
After Step 1	2	6	10
After Step 2	6	4	8
After Step 2	7	5	6
After Step 1	6	4	6
After Step 2	6	5	5
After Step 1	6	4	4
After Step 2	5	5	4
After Step 1	4	4	4

(b) Dave, Yona and Tam begin with 16, 0 and 0 candies, respectively.
The results of each step of the procedure are shown in the table. (We ignore Step 1 when each of the students has an even number of candies.)
Each student has 4 candies when the procedure ends.

	Dave	Yona	Tam
Start	16	0	0
After Step 2	8	8	0
After Step 2	4	8	4
After Step 2	4	6	6
After Step 2	5	5	6
After Step 1	4	4	6
After Step 2	5	4	5
After Step 1	4	4	4

(c) We begin by investigating the result that Step 2 has on a student's number of candies.
Assume Yona has c candies, and Dave (from whom Yona receives candies), has d candies.
Further, assume that c and d are both even integers.
During Step 2, Yona will give half of her candies away, leaving her with $\frac{c}{2}$ candies.

In this same Step 2, Yona will also receive $\frac{d}{2}$ candies from Dave (one half of Dave's d candies).

Therefore, Yona completes Step 2 with $\frac{c}{2} + \frac{d}{2} = \frac{c+d}{2}$ candies, which is the average of the c and d candies that Yona and Dave respectively began the step with.

On Wednesday, Dave starts with $2n$ candies, and each of Yona and Tam starts with $2n+3$

candies.

Since $2n+3$ is 3 more than a multiple of 2, then $2n+3$ is an odd integer for any integer n.

That is, we begin the procedure by performing Step 1 which leaves Dave with $2n$ candies ($2n$ is even and so no candies are discarded), and each of Yona and Tam with $2n+2$ candies.

After Step 2, Yona will have the average of her number of candies, $2n+2$, and Dave's number of candies, $2n$, or $\dfrac{(2n+2)+2n}{2} = \dfrac{4n+2}{2} = 2n+1$.

Tam will have the average of his number of candies, $2n+2$, and Yona's number of candies, $2n+2$, which is $2n+2$.

Dave will have the average of his number of candies, $2n$, and Tam's number of candies, $2n+2$, or $\dfrac{(2n+2)+2n}{2} = \dfrac{4n+2}{2} = 2n+1$.

Since Yona and Dave each now have an odd number of candies, Step 1 is performed.

The procedure is continued in the table shown.

	Dave	Yona	Tam
Start	$2n$	$2n+3$	$2n+3$
After Step 1	$2n$	$2n+2$	$2n+2$
After Step 2	$2n+1$	$2n+1$	$2n+2$
After Step 1	$2n$	$2n$	$2n+2$
After Step 2	$2n+1$	$2n$	$2n+1$
After Step 1	$2n$	$2n$	$2n$

At the end of the procedure, each student has $2n$ candies.

(d) On Thursday, Dave begins with 2^{2017} candies, gives one half or $\frac{1}{2} \times 2^{2017} = 2^{2016}$ to Yona, receives 0 from Tam, and thus completes the first Step 2 having 2^{2016} candies.

In the table shown, we proceed with the first few steps to get a sense of what is happening early in the procedure. (We again ignore Step 1 when each student has an even number of candies.)

	Dave	Yona	Tam
Start	2^{2017}	0	0
After Step 2	2^{2016}	2^{2016}	0
After Step 2	2^{2015}	2^{2016}	2^{2015}
After Step 2	2^{2015}	$2^{2014}+2^{2015}$	$2^{2015}+2^{2014}$
		$= 2^{2014} + 2 \times 2^{2014}$	$= 2 \times 2^{2014} + 2^{2014}$
		$= 3 \times 2^{2014}$	$= 3 \times 2^{2014}$
After Step 2	$2^{2015}+2^{2013}$	$2^{2013}+2^{2015}$	$2^{2015}+2^{2014}$
	$= 2^2 \times 2^{2013} + 2^{2013}$	$= 2^{2013} + 2^2 \times 2^{2013}$	$= 2 \times 2^{2014} + 2^{2014}$
	$= 5 \times 2^{2013}$	$= 5 \times 2^{2013}$	$= 3 \times 2^{2014}$

As was demonstrated in part (c), each application of Step 2 gives the average number of candies that two students had prior to the step.

If each of the three students has a number of candies that is divisible by 2^k for some positive integer k, then after performing Step 2, each student will have a number of candies that is divisible by 2^{k-1}. Why?

If Yona has a candies and Dave has b candies, where both a and b are divisible by 2^k, then after Step 2, Yona's number of candies is the average $\dfrac{a+b}{2} = \dfrac{a}{2} + \dfrac{b}{2}$.

Since a is divisible by 2^k, then $\dfrac{a}{2}$ is divisible by 2^{k-1} and similarly $\dfrac{b}{2}$ is divisible by 2^{k-1} and so their sum is at least divisible by 2^{k-1} (and possibly more).

We proceed by introducing 4 important facts which will lead us to our conclusion.

Important Fact #1:
We are starting with 2^{2017}, 0 and 0 candies, each of which is divisible by 2^{2017}.
The first application of Step 2 gives three numbers, each of which is divisible by 2^{2016}.
The second application of Step 2 gives three numbers, each of which is divisible by 2^{2015}.
The third application of Step 2 gives three numbers, each of which is divisible by 2^{2014}, and so on. (We can verify this in the table above.)
That is, starting with 2^{2017}, 0 and 0 candies, we are able to apply Step 2 2017 times in a row.
We note that at each of these 2017 steps, the number of candies that each student has is even, and therefore Step 1 is never applied (no candies have been discarded), and so the total number of candies shared by the three students is still 2^{2017}.

Important Fact #2:
If we begin Step 2 with $2a, 2a$ and $2b$ candies (exactly two students having an equal number of candies), then the result after applying Step 2 is $a + b, 2a$, and $a + b$ candies.
That is, there are still exactly two students who have an equal number of candies.

Important Fact #3:
If we begin Step 2 with $2a, 2a$ and $2b$ candies where $a < b$, then we call this a "2 low, 1 high state" (the two equal numbers are less than the third).
Applying Step 2 to $2a, 2a$ and $2b$ (a "2 low, 1 high state"), gives $a+b, 2a$, and $a+b$ which is a "2 high, 1 low state". (Since $a < b$, then $a + a < b + a$ or $2a < a + b$.)
Similarly, applying Step 2 again to this "2 high, 1 low state" gives a "2 low, 1 high state".
Since we begin with 2^{2017}, 0 and 0 candies, which is a "2 low, 1 high state", then after 2017 applications of Step 2, we will be at a "2 high, 1 low state".

Important Fact #4:
Beginning with $2a, 2a$ and $2b$ candies, the positive difference between the high number of candies and the low number of candies is $2b - 2a$ (or $2a - 2b$ if $a > b$).
After applying Step 2, we have $a + b, 2a$, and $a + b$ candies and the positive difference between the high and low numbers of candies is $b - a$ (or $a - b$ if $a > b$).
That is, applying Step 2 once decreases the positive difference between the high and low numbers of candies by a factor of 2 (that is, $a - b = \frac{1}{2}(2a - 2b)$).
Therefore, beginning with 2^{2017}, 0 and 0 candies, whose positive difference is 2^{2017}, and applying Step 2 2017 times gives a "2 high, 1 low state" where the positive difference between the high and low numbers is 1.
That is, after applying Step 2 2017 times, the number of candies is $n + 1, n + 1$ and n for some non-negative integer n.

Conclusion:
Since we haven't applied Step 1, then there are still 2^{2017} candies shared between the three students.
If n is odd, then the number of candies, $3n + 2$, is odd.
Since $3n + 2$ is equal to 2^{2017}, this is not possible and so n is even.
Since n is even, then $n + 1$ is odd and so we apply Step 1 to $n + 1, n + 1$ and n candies so that each student has an equal number of candies, n.
Two candies were discarded in the application of Step 1 and so there are now $2^{2017} - 2$ candies remaining.
Since each student has an equal number of candies, and there are $2^{2017} - 2$ candies in total, the procedure ends with each student having $\dfrac{2^{2017} - 2}{3}$ candies.