The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
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2017 Fermat Contest

(Grade 11)

Tuesday, February 28, 2017
(in North America and South America)

Wednesday, March 1, 2017
(outside of North America and South America)

Solutions

1. Evaluating, $6 \times 2017 - 2017 \times 4 = 2017(6-4) = 2017(2) = 4034$.

   Answer: (D)

2. In the diagram, there are 7 rows in which there are shaded squares, and there are 7 shaded squares in each row.
   Thus, there are $7 \cdot 7 = 49$ shaded squares.

   Answer: (E)

3. The sum of 2, 3 and 6 is $2+3+6=11$. Their product is $2 \cdot 3 \cdot 6 = 36$.

   Answer: (C)

4. Since 300 litres drains in 25 hours, then the rate at which water is leaving the tank equals $\dfrac{300\text{ L}}{25\text{ h}}$ or 12 L/h.

   Answer: (A)

5. The graph of $y = -2x^2 + 4$ is a parabola.
   Since the coefficient of x^2 is negative, the parabola opens downwards.
   Since the constant term is positive, the y-intercept of the parabola (that is, the value of y when $x = 0$) is positive.
   Of the given graphs, only (D) has these properties. (Since the coefficient of x is 0 in the given equation, then the graph should be symmetric about the y-axis, as the graph in (D) is.)

   Answer: (D)

6. Since the average of 5 and 9 is $\dfrac{5+9}{2} = 7$, then the averages of 5 and x and of x and 9 must be 10 and 12.
   In other words, $\dfrac{5+x}{2}$ and $\dfrac{x+9}{2}$ are equal to 10 and 12 in some order.
   Adding these, we obtain $\dfrac{5+x}{2} + \dfrac{x+9}{2} = 10 + 12$ or $\dfrac{14+2x}{2} = 22$ and so $7 + x = 22$ or $x = 15$.
   (We could have also noted that $\dfrac{5+x}{2} < \dfrac{x+9}{2}$ since $5 < 9$, and so $\dfrac{x+9}{2} = 12$.)

   Answer: (B)

7. Since $x = 1$ is a solution of the equation $x^2 + ax + 1 = 0$, then $1^2 + a(1) + 1 = 0$ or $2 + a = 0$ and so $a = -2$.

   Answer: (E)

8. Since $\dfrac{1}{2n} + \dfrac{1}{4n} = \dfrac{2}{4n} + \dfrac{1}{4n} = \dfrac{3}{4n}$, then the given equation becomes $\dfrac{3}{4n} = \dfrac{3}{12}$ or $4n = 12$.
   Thus, $n = 3$.

   Answer: (E)

9. We need to determine the time 100 hours before 5 p.m. Friday.
Since there are 24 hours in 1 day and since $100 = 4(24) + 4$, then 100 hours is equal to 4 days plus 4 hours.
Starting at 5 p.m. Friday, we move 4 hours back in time to 1 p.m. Friday and then an additional 4 days back in time to 1 p.m. Monday.
Thus, Kamile turned her computer on at 1 p.m. Monday.

ANSWER: (D)

10. Suppose that the integers $a < b < c < n$ have $a + b + c + n = 100$.
Since $a < b < c < n$, then $a + b + c + n < n + n + n + n = 4n$. Thus, $100 < 4n$ and so $n > 25$.
Since n is an integer, then n is at least 26.
Could n be 26? In this case, we would have $a + b + c = 100 - 26 = 74$.
If $n = 26$, then $a + b + c$ is at most $23 + 24 + 25 = 72$, which means that we cannot have $a + b + c = 74$.
Therefore, n cannot be 26.
Could n be 27? In this case, we would have $a + b + c = 100 - 27 = 73$.
Here, we could have $a+b+c = 23+24+26 = 73$, and so $n = 27$ is possible, which means that the smallest possible value of n is 27. (There are other values of a, b, c that work with $n = 27$ as well.)

ANSWER: (D)

11. Each student brought exactly one of an apple, a banana, and an orange.
Since 20% of the students brought an apple and 35% brought a banana, then the percentage of students who brought an orange is $100\% - 20\% - 35\% = 45\%$.
Therefore, the 9 students who brought an orange represent 45% of the class.
This means that 1 student represents $45\% \div 9 = 5\%$ of the class.
Thus, the class has $100\% \div 5\% = 20$ students in it.

ANSWER: (D)

12. The question is equivalent to asking how many three-digit positive integers beginning with 2 are larger than 217.
These integers are 218 through 299 inclusive.
There are $299 - 217 = 82$ such integers.

ANSWER: (B)

13. The line through $R(2, 4)$ and $Q(4, 0)$ has slope $\frac{4-0}{2-4} = -2$.
Since it passes through $(4, 0)$, this line has equation $y - 0 = -2(x - 4)$ or $y = -2x + 8$.
The line with equation $y = -2x + 8$ has y-intercept 8, and so the coordinates of P are $(0, 8)$.
Now, $\triangle OPQ$ is right-angled at O and so its area is $\frac{1}{2}(OQ)(OP) = \frac{1}{2}(4)(8) = 16$.

ANSWER: (E)

14. The expression

$$\left(1 + \tfrac{1}{2}\right)\left(1 + \tfrac{1}{3}\right)\left(1 + \tfrac{1}{4}\right)\left(1 + \tfrac{1}{5}\right)\left(1 + \tfrac{1}{6}\right)\left(1 + \tfrac{1}{7}\right)\left(1 + \tfrac{1}{8}\right)\left(1 + \tfrac{1}{9}\right)$$

is equal to

$$\left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{7}{6}\right)\left(\tfrac{8}{7}\right)\left(\tfrac{9}{8}\right)\left(\tfrac{10}{9}\right)$$

which equals

$$\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}$$

Removing common factors from the numerator and denominator, we obtain $\frac{10}{2}$ or 5.

ANSWER: (A)

15. Since $\angle XMZ = 30^\circ$, then $\angle XMY = 180^\circ - \angle XMZ = 180^\circ - 30^\circ = 150^\circ$.
Since the angles in $\triangle XMY$ add to 180°, then

$$\angle YXM = 180^\circ - \angle XYZ - \angle XMY = 180^\circ - 15^\circ - 150^\circ = 15^\circ$$

(Alternatively, since $\angle XMZ$ is an exterior angle of $\triangle XMY$, then $\angle XMZ = \angle YXM + \angle XYM$ which also gives $\angle YXM = 15^\circ$.)
Since $\angle XYM = \angle YXM$, then $\triangle XMY$ is isosceles with $MX = MY$.
But M is the midpoint of YZ, and so $MY = MZ$.
Since $MX = MY$ and $MY = MZ$, then $MX = MZ$.
This means that $\triangle XMZ$ is isosceles with $\angle XZM = \angle ZXM$.
Therefore, $\angle XZY = \angle XZM = \frac{1}{2}(180^\circ - \angle XMZ) = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$.

ANSWER: (A)

16. Since $x + 2y = 30$, then

$$\begin{aligned}
\frac{x}{5} + \frac{2y}{3} + \frac{2y}{5} + \frac{x}{3} &= \frac{x}{5} + \frac{2y}{5} + \frac{x}{3} + \frac{2y}{3} \\
&= \frac{1}{5}x + \frac{1}{5}(2y) + \frac{1}{3}x + \frac{1}{3}(2y) \\
&= \frac{1}{5}(x + 2y) + \frac{1}{3}(x + 2y) \\
&= \frac{1}{5}(30) + \frac{1}{3}(30) \\
&= 6 + 10 \\
&= 16
\end{aligned}$$

ANSWER: (B)

17. Suppose that the base of the prism is b cm by w cm and the height of the prism is h cm.
Since Aaron has 144 cubes with edge length 1 cm, then the volume of the prism is 144 cm^3, and so $bwh = 144$.
Since the perimeter of the base is 20 cm, then $2b + 2w = 20$ or $b + w = 10$.
Since b and w are positive integers, then we can make a chart of the possible combinations of b and w and the resulting values of $h = \dfrac{144}{bw}$, noting that since b and w are symmetric, then we can assume that $b \leq w$:

b	w	h
1	9	16
2	8	9
3	7	$\frac{48}{7}$
4	6	6
5	5	$\frac{144}{25}$

Since h must itself be a positive integer, then the possible values of h are 16, 9 and 6.
The sum of the possible heights is 16 cm + 9 cm + 6 cm = 31 cm.

ANSWER: (A)

18. For any positive real number x, $\lfloor x \rfloor$ equals the largest integer less than or equal to x and so $\lfloor x \rfloor \leq x$.
In particular, $\lfloor x \rfloor \cdot x \leq x \cdot x = x^2$.
Thus, if $\lfloor x \rfloor \cdot x = 36$, then $36 \leq x^2$.
Since $x > 0$, then $x \geq 6$.
In fact, if $x = 6$, then $\lfloor x \rfloor = \lfloor 6 \rfloor = 6$ and so $\lfloor x \rfloor \cdot x = x^2 = 36$. Therefore, $x = 6$. (Note that if $x > 6$, then $\lfloor x \rfloor \cdot x > 6 \cdot 6 = 36$.)
Also, since $\lfloor y \rfloor \cdot y = 71$, then $y^2 \geq 71$.
Since $y > 0$, then $y \geq \sqrt{71} \approx 8.43$.
Since $y \geq \sqrt{71} \approx 8.43$, then $\lfloor y \rfloor \geq 8$.
Suppose that $\lfloor y \rfloor = 8$.
In this case, $y = \dfrac{71}{\lfloor y \rfloor} = \dfrac{71}{8} = 8.875$. Note that if $y = \dfrac{71}{8}$, then $\lfloor y \rfloor = 8$, so $y = \dfrac{71}{8}$ is a solution.
(In fact, it is the only solution with $y > 0$. Can you see why?)
Therefore, $x + y = 6 + \dfrac{71}{8} = \dfrac{119}{8}$.

ANSWER: (B)

19. If $a > 0$, the distance from the vertical line with equation $x = a$ to the y-axis is a.
If $a < 0$, the distance from the vertical line with equation $x = a$ to the y-axis is $-a$.
In each case, there are exactly two points on the vertical line with equation $x = a$ that are also a distance of a or $-a$ (as appropriate) from the x-axis: (a, a) and $(a, -a)$. These points lie on the horizontal lines with equations $y = a$ and $y = -a$, respectively.
(If $a = 0$, the line $x = a$ coincides with the y-axis and the unique point on this line that is equidistant from the coordinate axes is the origin $(0, 0)$ which does not lie on the line with equation $3x + 8y = 24$.)
If the point (a, a) lies on the line $3x + 8y = 24$, then $3a + 8a = 24$ or $a = \frac{24}{11}$.
If the point $(a, -a)$ lies on the line $3x + 8y = 24$, then $3a - 8a = 24$ or $a = -\frac{24}{5}$.
The sum of these values of a is $\frac{24}{11} + \left(-\frac{24}{5}\right) = \frac{120-264}{55} = -\frac{144}{55}$.

ANSWER: (B)

20. Since m and n are positive integers with $n > 1$ and $m^n = 2^{25} \times 3^{40}$, then 2 and 3 are prime factors of m (since they are prime factors of m^n) and must be the only prime factors of m (since if there were other prime factors of m, then there would be other prime factors of m^n).
Therefore, $m = 2^a \times 3^b$ for some positive integers a and b and so $m^n = (2^a \times 3^b)^n = 2^{an} \times 3^{bn}$.
Since $m^n = 2^{25} \times 3^{40}$, then we must have $an = 25$ and $bn = 40$.
Since a, b, n are positive integers, then n is a common divisor of 25 and 40.
Since $n > 1$, then $n = 5$, which means that $a = 5$ and $b = 8$.
In this case, $m = 2^5 \times 3^8 = 32 \times 6561 = 209\,952$, which gives $m + n = 209\,952 + 5 = 209\,957$.

ANSWER: (C)

21. Since $WXYZ$ is a four-digit positive integer, then $WXYZ \leq 9999$. (In fact $WXYZ$ cannot be this large since all of its digits must be different.)
Since $WXYZ \leq 9999$, then $TWUYV \leq 2(9999) = 19998$.
Since $T \neq 0$, then $T = 1$.
Next, we note that the "carry" from any column to the next cannot be larger than 1. (Since $Z \leq 9$, then $Z + Z \leq 18$ and so the carry from the ones column to the tens column is 0 or 1. Similarly, since $Y + Y \leq 18$, then the largest sum of the digits plus carry in the tens column is 19 and so the maximum carry to the hundreds column is 1. This reasoning continues in the columns to the left.)
Thus, we make a chart of possible digits d and the resulting units digit in the sum from $d + d$ with and without a carry of 1:

d	Units digit of $d + d$ with no carry	Units digit of $d + d$ with carry of 1
0	0	1
1	2	3
2	4	5
3	6	7
4	8	9
5	0	1
6	2	3
7	4	5
8	6	7
9	8	9

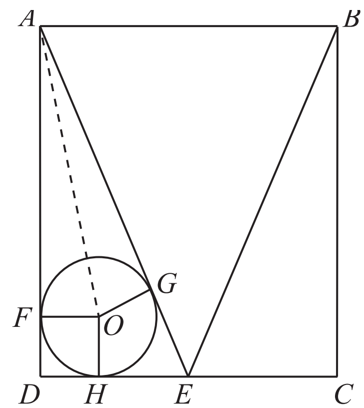
We use this table to first determine the digits W and Y.
Since the digits in the thousands column are all the same, then the digit W must be 9, since it must be at least 5 to produce a carry to the ten thousands column. We note further that this means that $X \geq 5$ to produce a carry into this column.
Also, the digit Y must equal 0 (since the digits T, U, V, W, X, Y, Z are different).
This means that there is no carry from the ones column to the tens column.
We summarize what we know so far:

$$\begin{array}{rccccc} & & 9 & X & 0 & Z \\ + & & 9 & X & 0 & Z \\ \hline & 1 & 9 & U & 0 & V \end{array}$$

and $X \geq 5$ and $Z \leq 4$.
Since $T = 1$ and $W = 9$, then Z can be 2, 3 or 4, and X can be 5, 6, 7, or 8.
Note that if $X = 5$, then we have $U = 0 = Y$, which is not possible, so $X \neq 5$.
If $Z = 2$, then $V = 4$. In this case, we cannot have $X = 6$ (which would give $U = 2 = Z$) or $X = 7$ (which would give $U = 4 = V$) and so $X = 8$, which gives $U = 6$.
If $Z = 3$, then $V = 6$. In this case, X cannot equal 6 or 8 and so $X = 7$ (which gives $U = 4$).
If $Z = 4$, then $V = 8$. In this case, X cannot equal 7 or 8 and so $X = 6$ (which gives $U = 2$).
In summary, there are 3 possible values for U, namely, 2, 4 and 6.
We can check that the sums $9802 + 9802 = 19604$ and $9703 + 9703 = 19406$ and $9604 + 9604 = 19208$ all satisfy the original problem.

Answer: (C)

22. We slice the cylinder, cone and sphere using a vertical plane that passes through the centres of the top and bottom faces of the cylinder and through the centre of the sphere.
The resulting cross-sections of the cylinder, cone and sphere are a rectangle, triangle and circle, respectively.
Since the sphere is touching the cylinder and the cone, then by slicing the cylinder in this way, the resulting circle is tangent to two sides of the rectangle (at F and H) and a side of the triangle (at G).



Join O to F, G and H. Since radii are perpendicular to tangents at the resulting points of tangency, then OF is perpendicular to AD, OG is perpendicular to AE, and OH is perpendicular to DE.
Let the radius of the sphere (now a circle) be r. Then $OF = OG = OH = r$.
Since the radius of the cylinder is 12, then $DE = 12$.
Since the height of the cylinder is 30, then $AD = 30$.
Since $FOHD$ has right angles at F, D and H, then it must have four right angles, and so is a rectangle.
Since $OF = OH = r$, then $FOHD$ is in fact a square with $DH = DF = r$.
Since $DE = 12$ and $DH = r$, then $EH = 12 - r$.
Since $AD = 30$ and $DF = r$, then $AF = 30 - r$.
Since AG and AF are tangents to the circle from the same point, then $AG = AF = 30 - r$.
(To see this, note that $\triangle AFO$ and $\triangle AGO$ are both right-angled, have a common side AO and equal sides FO and GO, which means that they are congruent.)
Similarly, $EG = EH = 12 - r$.
Finally, $AE = AG + GE$.
By the Pythagorean Theorem in $\triangle ADE$, $AE = \sqrt{12^2 + 30^2} = \sqrt{1044}$.
Thus, $\sqrt{1044} = (30 - r) + (12 - r)$ and so $2r = 42 - \sqrt{1044}$ or $r = 21 - \frac{1}{2}\sqrt{1044} \approx 4.8445$.
Of the given choices, this is closest to 4.84.

ANSWER: (A)

23. Since a is a positive integer and $a + \dfrac{b}{c}$ is a positive integer, then $\dfrac{b}{c}$ is a positive integer. In other words, b is a multiple of c.
Similarly, since $\dfrac{a}{c} + b$ is a positive integer and b is a positive integer, then a is a multiple of c.
Thus, we can write $a = Ac$ and $b = Bc$ for some positive integers A and B.
Therefore, $a + \dfrac{b}{c} = 101$ becomes $Ac + B = 101$ and $\dfrac{a}{c} + b = 68$ becomes $A + Bc = 68$.
Adding these new equations gives $Ac + B + A + Bc = 101 + 68$ or $A(c+1) + B(c+1) = 169$ and so $(A + B)(c + 1) = 169$.
Since $(A + B)(c + 1) = 169$, then $c + 1$ is a divisor of 169.
Since $169 = 13^2$, then the positive divisors of 169 are 1, 13, 169.
Since A, B, c are positive integers, then $A + B \geq 2$ and $c + 1 \geq 2$.
Since neither $A + B$ nor $c + 1$ can equal 1, then $A + B = c + 1 = 13$.
Finally, $\dfrac{a+b}{c} = \dfrac{Ac + Bc}{c} = A + B = 13$ and so $k = 13$.

ANSWER: (A)

24. We label the 8 teams as F, G, H, J, K, L, M, N.
We first determine the total number of games played.
Since each pair of teams plays exactly one game, then each team plays 7 games (one against each of the other 7 teams). Since there are 8 teams, then it seems as if there are $8 \cdot 7 = 56$ games, except that each game has been counted twice in this total (since, for example, we have counted G playing K and K playing G). Thus, there are in fact $\frac{8 \cdot 7}{2} = 28$ games played.
Since there are 28 games played and there are 2 equally likely outcomes for each game, then there are 2^{28} possible combinations of outcomes. (We can consider that the games are numbered from 1 to 28 and that F plays G in game 1, F plays H in game 2, and so on. A possible combination of outcomes for the tournament can be thought of as a "word" with 28 letters, the first letter being F or G (depending on the winner of the first game), the second letter being F or H (depending on the winner of the second game), and so on. There are two choices for each letter in the word, and so 2^{28} possible words.)
To determine the probability that every team loses at least one game and every team wins at least one game, we determine the probability that there is a team that loses 0 games or a team that wins 0 games and subtract this probability from 1.
Since we know the total number of possible combinations of outcomes, we determine the probability by counting the number of combinations of outcomes in which there is a team that loses 0 games (that is, wins all of its games) or a team that wins 0 games (that is, loses all of its games), or both.
To determine the number of combinations of outcomes in which there is a team that wins all of its games, we note that there are 8 ways to choose this team. Once a team is chosen (we call this team X), the results of the 7 games played by X are determined (X wins all of these) and the outcomes of the remaining $28 - 7 = 21$ games are undetermined.
Since there are two possible outcomes for each of these 21 undetermined games, then there are $8 \cdot 2^{21}$ combinations of outcomes in which there is a team that wins all of its games. (Note that there cannot be two teams that win all of their games, since these two teams have to play a game.) Similarly, there are $8 \cdot 2^{21}$ combinations of outcomes in which there is a team that loses all of its games. (Can you see why?)
Before arriving at our conclusion, we note that there might be combinations of outcomes that are included in both of these counts. That is, there might be combinations of outcomes in which there is a team that wins all of its games *and* in which there is a team that loses all of its games.
Since this total has been included in both sets of $8 \cdot 2^{21}$ combinations of outcomes, we need to determine this total and subtract it once to leave these combinations included exactly once in our total.
To determine the number of combinations of outcomes in this case, we choose a team (X) to win all of its games and a team (Y) to lose all of its games.
Once X is chosen, the outcomes of its 7 games are all determined (X wins).
Once Y is chosen, the outcomes of its 6 additional games are all determined (Y loses these 6 games plus the game with X that has already been determined).
The outcomes of the remaining $28 - 7 - 6 = 15$ games are undetermined.
Therefore, the number of combinations of outcomes is $8 \cdot 7 \cdot 2^{15}$ since there are 8 ways of choosing X, and then 7 ways of choosing Y (any team but X), and then 2^{15} combinations of outcomes for the undetermined games.
Thus, there are $8 \cdot 2^{21} + 8 \cdot 2^{21} - 8 \cdot 7 \cdot 2^{15}$ combinations of outcomes in which either one team loses 0 games or one team wins 0 games (or both).

Therefore, the probability that one team loses 0 games or one team wins 0 games is

$$\frac{8\cdot 2^{21}+8\cdot 2^{21}-8\cdot 7\cdot 2^{15}}{2^{28}}=\frac{2^{15}(8\cdot 2^{6}+8\cdot 2^{6}-8\cdot 7)}{2^{28}}=\frac{2^{3}\cdot 2^{6}+2^{3}\cdot 2^{6}-2^{3}\cdot 7}{2^{13}}=\frac{2^{6}+2^{6}-7}{2^{10}}$$

This means that the probability that every team loses at least one game and wins at least one game is $1-\frac{64+64-7}{1024}=1-\frac{121}{1024}=\frac{903}{1024}$.

Answer: (D)

25. Suppose that $r=\sqrt{\frac{\sqrt{53}}{2}+\frac{3}{2}}$.
Thus, $r^2=\frac{\sqrt{53}}{2}+\frac{3}{2}$ and so $2r^2=\sqrt{53}+3$ or $2r^2-3=\sqrt{53}$.
Squaring both sides again, we obtain $(2r^2-3)^2=53$ or $4r^4-12r^2+9=53$ which gives $4r^4-12r^2-44=0$ or $r^4-3r^2-11=0$ or $r^4=3r^2+11$.

Suppose next that

$$r^{100}=2r^{98}+14r^{96}+11r^{94}-r^{50}+ar^{46}+br^{44}+cr^{40} \qquad (*)$$

for some positive integers a,b,c.
Since $r\neq 0$, we can divide by r^{40} to obtain

$$r^{60}=2r^{58}+14r^{56}+11r^{54}-r^{10}+ar^{6}+br^{4}+c$$

Now using the relationship $r^4=3r^2+11$, we can see that

$$\begin{aligned}r^{60}-2r^{58}-14r^{56}-11r^{54}&=r^{54}(r^6-2r^4-14r^2-11)\\&=r^{54}(r^2(3r^2+11)-2r^4-14r^2-11)\\&=r^{54}(3r^4+11r^2-2r^4-14r^2-11)\\&=r^{54}(r^4-3r^2-11)\\&=r^{54}(0)\\&=0\end{aligned}$$

Therefore, the equation $(*)$ is equivalent to the much simpler equation

$$r^{10}=ar^6+br^4+c$$

Next, we express r^{10} and r^6 as combinations of r^2 and constant terms. (To do this, we will need to express r^8 in this way too.)

$$r^6=r^2r^4=r^2(3r^2+11)=3r^4+11r^2=3(3r^2+11)+11r^2=20r^2+33$$
$$r^8=r^2r^6=r^2(20r^2+33)=20r^4+33r^2=20(3r^2+11)+33r^2=93r^2+220$$
$$r^{10}=r^2r^8=r^2(93r^2+220)=93r^4+220r^2=93(3r^2+11)+220r^2=499r^2+1023$$

Therefore, the equation

$$r^{10}=ar^6+br^4+c$$

is equivalent to

$$499r^2+1023=a(20r^2+33)+b(3r^2+11)+c$$

Rearranging, we obtain

$$0=r^2(20a+3b-499)+(33a+11b+c-1023)$$

Therefore, if $20a + 3b = 499$ and $33a + 11b + c = 1023$, then the equation is satisfied. (It also turns out that if the equation is satisfied, then it must be the case that $20a + 3b = 499$ and $33a + 11b + c = 1023$. This is because r^2 is an irrational number.)

So the original problem is equivalent to finding positive integers a, b, c with $20a + 3b = 499$ and $33a + 11b + c = 1023$.

We proceed by finding pairs (a, b) of positive integers that satisfy $20a + 3b = 499$ and then checking to see if the value of $c = 1023 - 33a - 11b$ is positive. Since we need to find one triple (a, b, c) of positive integers, we do not have to worry greatly about justifying that we have all solutions at any step.

Since $20a$ has a ones digit of 0 and $20a + 3b = 499$, then the ones digit of $3b$ must be 9, which means that the ones digit of b must be 3.

If $b = 3$, we obtain $20a = 499 - 3b = 490$ and so a is not an integer.

If $b = 13$, we obtain $20a = 499 - 3b = 460$ and so $a = 23$.

Note that, from $(a, b) = (23, 13)$, we can obtain additional solutions by noticing that $20(3) = 3(20)$ and so if we decrease a by 3 and increase b by 20, the sum $20a + 3b$ does not change.

However, it turns out that if $(a, b) = (23, 13)$, then $c = 1023 - 33(23) - 11(13) = 121$.

Since we are only looking for a unique triple (a, b, c), then $(a, b, c) = (23, 13, 121)$.

Finally, $a^2 + b^2 + c^2 = 23^2 + 13^2 + 121^2 = 15\,339$.

ANSWER: (D)