The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

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Canadian Senior Mathematics Contest

Wednesday, November 22, 2017
(in North America and South America)

Thursday, November 23, 2017
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) previously stored information such as formulas, programs, notes, etc., (iv) a computer algebra system, (v) dynamic geometry software.

Do not open this booklet until instructed to do so.

There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the Web site, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

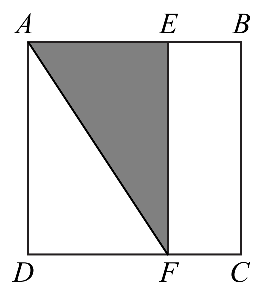
Useful Facts:
The following facts may be helpful:
- If n is a positive integer, the sum of the n integers from 1 to n equals $\frac{1}{2}n(n+1)$ (that is, $1 + 2 + 3 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$), and
- If n is a positive integer, the sum of the first n perfect squares equals $\frac{1}{6}n(n+1)(2n+1)$ (that is, $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{1}{6}n(n+1)(2n+1)$).

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. What is the value of $\frac{6}{3} \times \frac{9}{6} \times \frac{12}{9} \times \frac{15}{12}$?

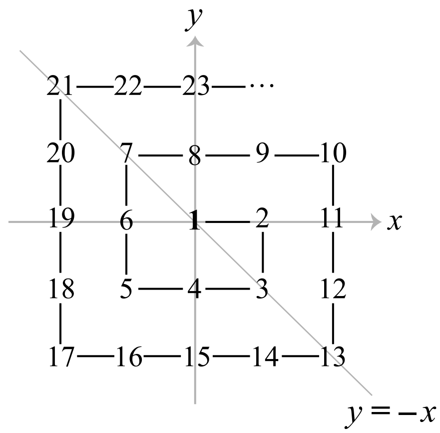
2. In the diagram, $ABCD$ is a square with side length 8 cm. Point E is on AB and point F is on DC so that $\triangle AEF$ is right-angled at E. If the area of $\triangle AEF$ is 30% of the area of $ABCD$, what is the length of AE?



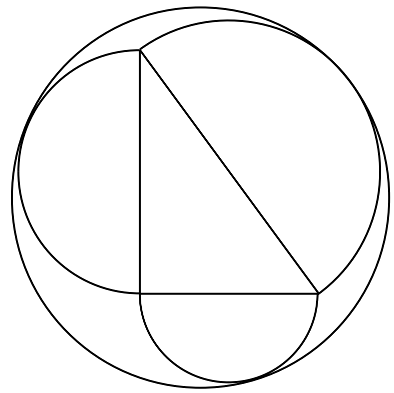
3. Determine all real numbers x for which $x^4 - 3x^3 + x^2 - 3x = 0$.

4. How many five-digit positive integers can be formed by arranging the digits $1, 1, 2, 3, 4$ so that the two 1s are not next to each other?

5. The rectangular spiral shown in the diagram is constructed as follows. Starting at $(0,0)$, line segments of lengths $1,1,2,2,3,3,4,4,\ldots$ are drawn in a clockwise manner, as shown. The integers from 1 to 1000 are placed, in increasing order, wherever the spiral passes through a point with integer coordinates (that is, 1 at $(0,0)$, 2 at $(1,0)$, 3 at $(1,-1)$, and so on). What is the sum of all of the positive integers from 1 to 1000 which are written at points on the line $y=-x$?



6. In the diagram, the triangle has side lengths 6, 8 and 10. Three semi-circles are drawn using the sides of the triangle as diameters. A large circle is drawn so that it just touches each of the three semi-circles. What is the radius of the large circle?



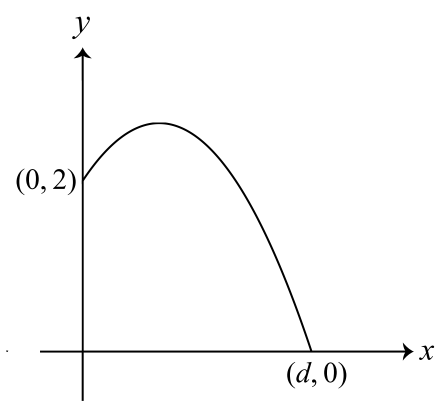
PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. (a) Determine all real numbers x for which $x^2+2x-8=0$.

   (b) Determine the values of b and c for which the parabola with equation $y=x^2+bx+c$ passes through the points $(1,2)$ and $(2,0)$.

   (c) A ball is thrown from a window at the point $(0,2)$. The ball flies through the air along the parabola with equation $y=a(x-1)^2+\frac{8}{3}$, for some real number a. Determine the positive real number d for which the ball hits the ground at $(d,0)$.



2. Joe plays a game using some cards, each of which is red on one side and green on the other side. To begin the game, Joe selects integers k and n with $1 \leq k < n$. He then lays n cards on the table in a row with the red sides all facing up. On each turn, Joe flips over exactly k of the n cards. Joe wins the game if, after an integer number of turns, he has flipped the cards so that each card has its green side facing up. For example, when $n = 4$ and $k = 3$, Joe can win the game in 4 turns, as shown.

R	R	R	R	Start
R	G	G	G	After 1 turn
G	G	R	R	After 2 turns
R	R	R	G	After 3 turns
G	G	G	G	After 4 turns

(a) When $n = 6$ and $k = 4$, show that Joe can win the game in 3 turns.

(b) When $n = 9$ and $k = 5$, show that Joe can win the game.

(c) Suppose that $n = 2017$. Determine, with justification, all integers k with $1 \leq k < 2017$ for which Joe cannot win the game.

3. (a) Consider a function f with $f(1) = 2$ and $f(f(n)) = f(n) + 3n$ for all positive integers n. When we substitute $n = 1$, the equation $f(f(n)) = f(n) + 3n$ becomes $f(f(1)) = f(1) + 3(1)$. Since $f(1) = 2$, then $f(2) = 2 + 3 = 5$. Continuing in this way, determine the value of $f(26)$.

(b) Prove that there is no function g with $g(1) = 2$ and $g(g(n) + m) = n + g(m)$ for all positive integers n and m.

(c) Prove that there is exactly one function h with the following properties:

- the domain of h is the set of positive integers,
- $h(n)$ is a positive integer for every positive integer n, and
- $h(h(n) + m) = 1 + n + h(m)$ for all positive integers n and m.