The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2016 Hypatia Contest

Wednesday, April 13, 2016
(in North America and South America)

Thursday, April 14, 2016
(outside of North America and South America)

Solutions

1. (a) Since 5 baskets of raisins fill 2 tubs, then $5 \times 6 = 30$ baskets of raisins fill $2 \times 6 = 12$ tubs. Therefore, 12 tubs of raisins fill 30 baskets.

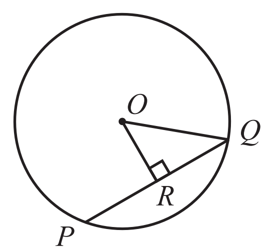
   (b) Since 5 scoops of raisins fill 1 jar, then $5 \times 6 = 30$ scoops of raisins fill $1 \times 6 = 6$ jars. Since 3 scoops of raisins fill 1 cup, then $3 \times 10 = 30$ scoops of raisins fill $1 \times 10 = 10$ cups. Since 30 scoops fill 6 jars, and 30 scoops fill 10 cups, then 10 cups of raisins fill 6 jars.

   (c) *Solution 1*
   From part (b), we know that 10 cups of raisins fill 6 jars.
   Thus, $10 \times 5 = 50$ cups of raisins fill $6 \times 5 = 30$ jars.
   Since 30 jars of raisins fill 1 tub, then 50 cups of raisins fill 1 tub, or $50 \times 2 = 100$ cups of raisins fill $1 \times 2 = 2$ tubs.
   Since 2 tubs of raisins fill 5 baskets, then 100 cups of raisins fill 5 baskets.
   This tells us that $100 \div 5 = 20$ cups of raisins fill $5 \div 5 = 1$ basket.

   Solution 2
   Since 5 baskets fill 2 tubs, then $\frac{2}{5}$ tubs fill 1 basket.
   Since 30 jars of raisins fill 1 tub, then $\frac{2}{5} \times 30 = 12$ jars of raisins fill $\frac{2}{5}$ tubs and so fill 1 basket.
   Since 5 scoops of raisins fill 1 jar, then $12 \times 5 = 60$ scoops of raisins fill 12 jars and so fill 1 basket.
   Since 3 scoops of raisins fill 1 cup, then $20 \times 1 = 20$ cups fill $20 \times 3 = 60$ scoops and so fill 1 basket.
   Therefore, 20 cups of raisins fill 1 basket.

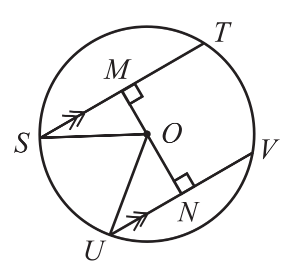
2. (a) Since M is the midpoint of chord AB, then $AM = \frac{1}{2}(AB) = 5$.
   Also, since M is the midpoint of chord AB, then OM is perpendicular to AB.
   Using the Pythagorean Theorem in $\triangle OMA$, we get $OM^2 = OA^2 - AM^2$ or $OM^2 = 13^2 - 5^2 = 169 - 25 = 144$, and so $OM = \sqrt{144} = 12$ (since $OM > 0$).

   (b) Let the circle have centre O and chord PQ, as shown.
   Since the radius is 25, then $OQ = 25$.
   The perpendicular distance from O to the chord is given by OR, and so $OR = 7$.
   In $\triangle ORQ$, the Pythagorean Theorem gives $RQ^2 = OQ^2 - OR^2$ or $RQ^2 = 25^2 - 7^2 = 625 - 49 = 576$, and so $RQ = \sqrt{576} = 24$ (since $RQ > 0$).



   Since OR is perpendicular to the chord PQ, then R is the midpoint of PQ, and so $PQ = 2(RQ) = 2(24) = 48$.
   Therefore, the length of the chord is 48.

   (c) Join O to S and O to U, as shown.
   The radius of the circle is 65, and so $OS = OU = 65$.
   Since OM is perpendicular to chord ST, then M is the midpoint of the chord and so $MS = \frac{1}{2}(ST) = \frac{1}{2}(112) = 56$.
   In $\triangle OMS$, the Pythagorean Theorem gives $OM^2 = OS^2 - MS^2$ or $OM^2 = 65^2 - 56^2 = 4225 - 3136 = 1089$, and so $OM = \sqrt{1089} = 33$ (since $OM > 0$).



   Since $MN = OM + ON = 72$, then $ON = 72 - OM = 72 - 33 = 39$.
   In $\triangle ONU$, the Pythagorean Theorem gives $NU^2 = OU^2 - ON^2$ or $NU^2 = 65^2 - 39^2 = 4225 - 1521 = 2704$, and so $NU = \sqrt{2704} = 52$ (since $NU > 0$).

Finally, since ON is perpendicular to chord UV, then N is the midpoint of the chord and so $UV = 2(NU) = 2(52) = 104$.
Therefore, the length of the chord UV is 104.

3. (a) Since $405 = 3^4 \times 5$, then 405 is divisible by 3^4 but is not divisible by 3^5.
Thus, $f(405) = 4$.

(b) First, we find all factors of 3 which exist in the product $1\times2\times3\times4\times5\times6\times7\times8\times9\times10$.
The multiples of 3 are the only numbers which contain factors of 3.
The multiples of 3 in the given product are 3, 6 and 9.
Rewriting the given product, we get

$$\begin{aligned} &1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &\quad = \quad 1 \times 2 \times 3 \times 4 \times 5 \times (2 \times 3) \times 7 \times 8 \times (3 \times 3) \times 10 \\ &\quad = \quad 3^4 \times (1 \times 2 \times 4 \times 5 \times 2 \times 7 \times 8 \times 10). \end{aligned}$$

Since the product in parentheses does not include any factors of 3, then the largest power of 3 which divides the given product is 3^4, and so $f(1\times2\times3\times4\times5\times6\times7\times8\times9\times10) = 4$.

(c) First, we count the number of factors of 3 included in 100!.
Every multiple of 3 includes least 1 factor of 3.
The product 100! includes 33 multiples of 3 (since $33 \times 3 = 99$).
Counting one factor of 3 from each of the multiples of 3 (these are $3, 6, 9, 12, 15, 18, \ldots, 93, 96, 99$), we see that 100! includes at least 33 factors of 3.
However, each multiple of $3^2 = 9$ includes a second factor of 3 (since $9 = 3^2, 18 = 3^2 \times 2$, etc.) which was not counted in the previous 33 factors.
The product 100! includes 11 multiples of 9 (since $11 \times 9 = 99$), and thus there are at least 11 additional factors of 3 in 100!.
Similarly, 100! includes 3 multiples of $3^3 = 27$, each of which contribute an additional factor of 3 (these are $27 = 3^3, 54 = 3^3 \times 2$, and $81 = 3^4$).
Finally, there is one multiple of $3^4 = 81$ which contributes one more factor of 3.
Since $3^5 > 100$, then 100! does not include any multiples of 3^5 and so we have counted all possible factors of 3.
Thus, 100! includes exactly $33 + 11 + 3 + 1 = 48$ factors of 3, and so $100! = 3^{48} \times t$ for some positive integer t that is not divisible by 3.
Counting in a similar way, the product 50! includes 16 multiples of 3, 5 multiples of 9, and 1 multiple of 27, and thus includes $16 + 5 + 1 = 22$ factors of 3.
Therefore, $50! = 3^{22} \times r$ for some positive integer r that is not divisible by 3.
Also, 20! includes $6 + 2 = 8$ factors of 3, and thus $20! = 3^8 \times s$ for some positive integer s that is not divisible by 3.
Therefore, $N = \dfrac{100!}{50!20!} = \dfrac{3^{48} \times t}{(3^{22} \times r)(3^8 \times s)} = \dfrac{3^{48} \times t}{(3^{30} \times rs)} = \dfrac{3^{18} \times t}{rs}$.
Since we are given that N is equal to a positive integer, then $\dfrac{3^{18} \times t}{rs}$ is a positive integer.
Since r and s contain no factors of 3 and $3^{18} \times t$ is divisible by rs, then it must be the case that t is divisible by rs.
In other words, we can re-write $N = \dfrac{3^{18} \times t}{rs}$ as $N = 3^{18} \times \dfrac{t}{rs}$ where $\dfrac{t}{rs}$ is an integer.
Since each of r, s and t does not include any factors of 3, then the integer $\dfrac{t}{rs}$ is not

divisible by 3.

Therefore, the largest power of 3 which divides $\dfrac{100!}{50!20!}$ is 3^{18}, and so $f(N) = 18$.

(d) Since $f(a) = 8$, then the exponent of the largest power of 3 that divides a is 8.
That is, $a = 3^8 m$ for some positive integer m and 3 does not divide m.
Since $f(b) = 7$, then the exponent of the largest power of 3 that divides b is 7.
That is, $b = 3^7 n$ for some positive integer n and 3 does not divide n.
Substituting and simplifying, we get

$$a + b = 3^8 m + 3^7 n = 3^7(3m + n)$$

Since 3 divides $3m$ but 3 does not divide n, then 3 does not divide the sum $3m + n$.
That is, $3m+n$ is not a multiple of 3 and so the largest power of 3 that divides $a+b$ is 3^7.
Therefore, $f(a + b) = 7$.

4. (a) (i) For every 10 cents that one restaurant's price is higher than the other restaurant's price, it loses one customer to the other restaurant.
On Monday, LP charges $\$9.30 - \$7.70 = \$1.60$ more per pizza than what EP charges.
Therefore, LP loses $\frac{1.60}{0.10} = 16$ customers to EP and thus has $50 - 16 = 34$ customers.

(ii) The cost for LP to make each pizza is \$5.00, and so LP's profit is $\$9.30-\$5.00 = \$4.30$ for each pizza sold.
On Monday, LP's total profit is $\$4.30 \times 34 = \$146.20$.

(b) *Solution 1*
Let LP's price per pizza on Tuesday be $\$L$, where $L > 0$ and L is an integer multiple of 0.10.
If LP charges $\$L$ per pizza, then its profit is $\$(L - 5)$ per pizza sold.
We note that if $L < 5$, then LP's profit per pizza sold is negative (that is, LP is losing money on each pizza it sells).
Since EP charges \$7.20 per pizza, then the number of customers that LP has is $50 + \dfrac{7.20 - L}{0.10}$.
We note that if $L < 7.20$ (LP charges less per pizza than EP charges), then $\dfrac{7.20 - L}{0.10} > 0$ and LP will have more than 50 customers. In fact, LP gains $\dfrac{7.20 - L}{0.10}$ customers.
Similarly, if $L > 7.20$ (LP charges more per pizza than EP charges), then $\dfrac{7.20 - L}{0.10} < 0$ and LP will have fewer than 50 customers. In fact, LP loses $\dfrac{L - 7.20}{0.10}$ customers.
LP's profit on Tuesday is given by the product of its number of customers and its profit per pizza sold.
That is, LP's profit in dollars, P, is $P = \left(50 + \dfrac{7.20 - L}{0.10}\right) \times (L - 5)$.
Simplifying, we get $P = \left(\dfrac{5 + 7.2 - L}{0.10}\right) \times (L - 5) = 10(12.2 - L)(L - 5)$.
Therefore, P is a quadratic function of L.
The graph of this quadratic function, $P = 10(12.2 - L)(L - 5)$, is a parabola opening downward and thus the maximum profit occurs at its vertex.
The zeros of this parabola occur when $12.2 - L = 0$ (that is, when $L = 12.2$) and when

$L-5=0$ (that is, when $L=5$).
The vertex of the parabola occurs on its axis of symmetry, which is the vertical line passing through the midpoint of its zeros, $L=12.2$ and $L=5$.
That is, the maximum profit occurs when $L=\dfrac{12.2+5}{2}=\dfrac{17.2}{2}=8.60$.
On Tuesday, LP should charge \$8.60 per pizza to maximize their profit.

Solution 2
On Tuesday, EP charges \$7.20 per pizza.
Suppose that, on Tuesday, LP charges $\$(7.20+0.10d)$ per pizza for some integer d. (Note that LP's price must be an integer multiple of 10 cents higher or lower than EP's price.)
If $d>0$, then LP will lose d customers to EP.
If $d<0$, then LP will gain $-d$ customers from EP.
In other words, on Tuesday, LP will have $50-d$ customers.
Since it costs LP \$5.00 to make each pizza, then LP's profit per pizza is equal to $\$(7.20+0.10d)-\$5.00=\$(2.20+0.10d)$.
Therefore, in dollars, LP's profit on Tuesday is the product of its number of customers and its profit per pizza sold, or $P=(2.20+0.10d)(50-d)=0.10(22+d)(50-d)$.
Therefore, P is a quadratic function of d.
The graph of this quadratic function, $P=0.10(22+d)(50-d)$, is a parabola opening downward and thus the maximum profit occurs at its vertex.
The zeros of this parabola occur when $22+d=0$ (that is, when $d=-22$) and when $50-d=0$ (that is, when $d=50$).
The vertex of the parabola occurs on its axis of symmetry, which is the vertical line passing through the midpoint of its zeros, $d=-22$ and $d=50$.
That is, the maximum profit occurs when $d=\dfrac{(-22)+50}{2}=14$.
On Tuesday, LP should charge $\$(7.20+0.10(14))=\$8.60$ per pizza to maximize their profit.

(c) *Solution 1*
Suppose that EP set its price per pizza at $\$E$, where $E>0$ and E is an integer multiple of 0.20.
After EP sets its price at $\$E$, LP maximizes its profit by setting its price per pizza at $\$L$, where $L>0$ and L is an integer multiple of 0.10.
Let EP's profit be P_E and LP's profit be P_L.
First we determine the price per pizza, $\$L$, that LP will choose in order to maximize its profit, P_L, given that LP knows that EP has set its price per pizza at $\$E$.
LP's profit per pizza sold is $\$(L-5)$ and, using a similar method as in (b), its number of customers is $50+\dfrac{E-L}{0.10}$.
Thus, LP's total profit, in dollars, is given by $P_L=\left(50+\dfrac{E-L}{0.10}\right)\times(L-5)$.
Simplifying, we get $P_L=\left(\dfrac{5+E-L}{0.10}\right)\times(L-5)=10(5+E-L)(L-5)$.
We think about E as fixed and L as variable, making this a quadratic function in L.
The graph of this quadratic function, $P_L=10(5+E-L)(L-5)$, is a parabola opening downward and thus the maximum profit occurs at its vertex.
The zeros of this parabola occur when $5+E-L=0$ (that is, $L=5+E$) and when $L-5=0$ (that is, $L=5$).

The vertex of the parabola occurs on its axis of symmetry, which is the vertical line passing through the midpoint of its zeros, $L = 5 + E$ and $L = 5$.
That is, the maximum profit for LP occurs when $L = \dfrac{5 + E + 5}{2} = \dfrac{10 + E}{2} = 5 + \frac{1}{2}E$.
(Since E is a multiple of 0.20, then L is a multiple of 0.10.)
Thus, if EP first sets its price per pizza at $\$E$, then LP should charge $\$(5 + \frac{1}{2}E)$ per pizza to maximize its profit.

Since EP realizes what LP is doing, we can assume that EP now knows that LP will set their price per pizza at $\$(5 + \frac{1}{2}E)$.
Thus, EP may determine its price per pizza, $\$E$, that will maximize its profit.
EP's profit per pizza sold is $\$(E - 5)$ and its number of customers is $50 + \dfrac{L - E}{0.10}$.
(Since L and E are both multiples of 0.10, then this number is an integer.)
Thus, EP's total profit is given by $P_E = \left(50 + \dfrac{L - E}{0.10}\right) \times (E - 5)$.
Simplifying, we get $P_E = \left(\dfrac{5 + L - E}{0.10}\right) \times (E - 5) = 10(5 + L - E)(E - 5)$.
Since $L = 5 + \frac{1}{2}E$, the quadratic function becomes $P_E = 10(5 + (5 + \frac{1}{2}E) - E)(E - 5)$, or $P_E = 10(10 - \frac{1}{2}E)(E - 5)$.
This is again a parabola opening downward and so its maximum profit occurs at its vertex.
The zeros of this parabola occur when $E = 20$ and when $E = 5$.
Thus, the maximum profit for EP occurs when $E = \dfrac{20 + 5}{2} = 12.50$.
However, since E must equal an integer multiple of 0.20, then E cannot equal \$12.50.
Since the quadratic relation P_E is quadratic in E and the resulting parabola opens downward, then values of E closest to the vertex give the largest values corresponding values of P_E.
Therefore, to maximize EP's profit, we choose the closest values to $E = 12.50$ that are multiples of 20 cents.
These values are $E = 12.40$ (which gives $L = 11.20$), and $E = 12.60$ (which gives $L = 11.30$).
We note that $E = 12.40$ and $E = 12.60$ are symmetric about the axis of symmetry, $E = 12.50$, and thus give equal values of $P_E = 281.20$. Further, there are no values of E which satisfy the given conditions and for which P_E is greater in value, since there are no multiples of 20 cents between \$12.40 and \$12.50 or between \$12.60 and \$12.50.
When EP sets its price at $E = 12.40$, LP's profit is $P_L = 10(5 + E - L)(L - 5)$ or $P_L = 10(5 + 12.40 - 11.20)(11.20 - 5) = 10(6.20)(6.20) = 384.40$.
When EP sets its price at $E = 12.60$, LP's profit is $P_L = 10(5 + E - L)(L - 5)$ or $P_L = 10(5 + 12.60 - 11.30)(11.30 - 5) = 10(6.30)(6.30) = 396.90$.

To maximize its profit, EP could charge \$12.40 or \$12.60 per pizza, which result in profits for LP of \$384.40 and \$396.90, respectively.

Solution 2
On Wednesday, suppose that EP charges $\$2e$ per pizza, where e is a multiple of 0.10.
Based on this fixed (but unknown) price, LP chooses its price on Wednesday to maximize its profit.
Suppose that, on Wednesday, LP charges $\$(2e + 0.10n)$ per pizza for some integer n. (Note that LP's price must be an integer multiple of 10 cents higher or lower than EP's price.)

As in (b), on Wednesday, LP will have $50 - n$ customers.
Since it costs LP \$5.00 to make each pizza, then LP's profit per pizza is equal to $\$(2e + 0.10n) - \$5.00 = \$(2e + 0.10n - 5)$.
Therefore, in dollars, LP's profit on Wednesday is

$$P_L = (2e+0.10n-5)(50-n) = 0.10(20e+n-50)(50-n) = -0.10n^2+(10-2e)n+(100e-250)$$

We treat e as a constant and n as a variable. Therefore, P_L is a quadratic function of n.
Since the coefficient of n^2 is negative, the graph of this quadratic function is a parabola opening downward and thus the maximum profit for LP occurs at its vertex.
The vertex occurs when $n = -\dfrac{10-2e}{2(-0.10)} = 50 - 10e$.
In this case, LP's profit, in dollars, is

$$P_L = 0.10(20e + (50 - 10e) - 50)(50 - (50 - 10e)) = 0.10(10e)(10e) = 10e^2$$

Now, on Wednesday, EP realizes what LP is doing and so sets its initial price, \$2e, to maximize EP's profit (knowing that LP will pick its price afterwards to optimize LP's profit).
Since EP's price is set at \$2e per pizza, then its profit per pizza is $\$(2e - 5)$.
Since LP has $50 - n$ customers and there are 100 customers in total, then EP has $100 - (50 - n) = 50 + n = 50 + (50 - 10e) = 100 - 10e$ customers. (From above, we can assume that $n = 50 - 10e$.)
Therefore, in dollars, EP's total profit on Wednesday is

$$P_E = (100 - 10e)(2e - 5) = -20e^2 + 250e - 500 = -20(e^2 - 12.5e + 25)$$

Completing the square, we obtain

$$P_E = -20((e - 6.25)^2 - 6.25^2 + 25) = -20(e - 6.25)^2 + 281.25$$

This is the equation of a parabola opening downwards. Thus, the maximum value of P_E occurs when $e = 6.25$. However, we require that e be a multiple of 0.10.
To find the maximum value(s) of P_E including this constraint, we take the closest values of e to the vertex that are multiples of 0.10. These are $e = 6.20$ and $e = 6.30$.
Since $e = 6.20$ and $e = 6.30$ are symmetric about the vertex $e = 6.25$, then they give the same profit P_E, namely $P_E = 281.20$. Since we have stayed as closed to the vertex as possible, this is EP's maximum possible profit given the constraints.
When $e = 6.20$, EP's price is \$12.40 and LP's profit is $\$10e^2 = \$10(6.20)^2 = \$384.40$.
When $e = 6.30$, EP's price is \$12.60 and LP's profit is $\$10e^2 = \$10(6.30)^2 = \$396.90$.
To maximize its profit, EP should charge \$12.40 or \$12.60 per pizza, which result in profits for LP of \$384.40 and \$396.90, respectively.