The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Wednesday, November 23, 2016
(in North America and South America)

Thursday, November 24, 2016
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Do not open this booklet until instructed to do so.

There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the Web site, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

PART A

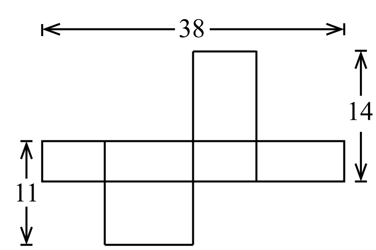
For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. What is the value of $\dfrac{1^3 + 2^3 + 3^3}{(1+2+3)^2}$?

2. Mike bought two books from the Canadian Super Mathematics Company. He paid full price for a \$33 book and he received 50% off the full price of a second book. In total, he saved 20% on his purchase. In dollars, how much did he save?

3. How many different lists a, b, c, d of distinct odd positive integers with $a < b < c < d$ have the property that $a + b + c + d = 24$?

4. The figure (called a *net*) has dimensions as shown. This net can be folded to form a rectangular prism. What is the volume of the prism?



5. Gary and Deep play a game in which there are no ties. Each player is equally likely to win each game. The first player to win 4 games becomes the champion, and no further games are played. Gary wins the first two games. What is the probability that Deep becomes the champion?

6. A nine-digit positive integer n is called a *Moffat number* if

   - its digits are $0, 1, 2, 3, 4, 5, 6, 7, 8$ with no repetition,
   - the sum of each group of 5 digits in a row is divisible by 5, and
   - the sum of each group of 7 digits in a row is divisible by 4.

   For example, 578460213 is *not* a Moffat number because, while the sum of each group of 7 digits in a row is divisible by 4, not all of the sums of groups of 5 digits in a row are divisible by 5. Determine the sum of all Moffat numbers.

PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. A positive integer that is the same when read forwards or backwards is called a *palindrome*. A table is created, as shown, in which the first column contains all palindromes between 1000 and 10 000, listed in increasing order. The second column contains the positive difference between each pair of consecutive palindromes. For example, 1001 and 1111 are consecutive palindromes and their positive difference is $1111 - 1001 = 110$.

Palindrome	Difference
1001	
	110
1111	
	110
1221	
	110
1331	
	110
1441	
⋮	⋮

   (a) What are the eighth and ninth palindromes in the first column of the table?

   (b) The second column of the table contains only two different numbers: 110 and x. Determine the value of x.

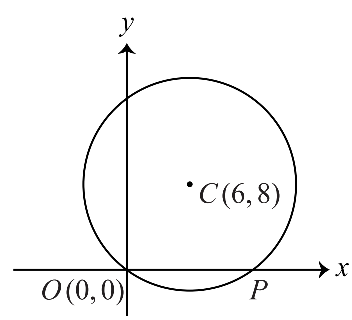
   (c) When the table is complete, there are N palindromes in the first column. What is the value of N?

   (d) Determine the average of all of the $N - 1$ numbers in the second column of the table.

2. In the diagram, the circle shown has centre $C(6, 8)$, has radius 10, and has equation

$$(x - 6)^2 + (y - 8)^2 = 100$$

   The circle passes through the origin $O(0, 0)$.



   (a) Determine the coordinates of the point P where the circle intersects the x-axis.

   (b) Determine, without justification, the coordinates of the point Q on the circle that has the maximum y-coordinate among all points on the circle.

   (c) Determine the coordinates of the point R on the circle so that $\angle PQR = 90°$.

   (d) Determine the coordinates of the two distinct points S and T on the circle so that $\angle PQS = \angle PQT = 45°$.

3. For each positive integer n, define a_n and b_n to be the positive integers such that

$$(\sqrt{3}+\sqrt{2})^{2n} = a_n + b_n\sqrt{6} \quad \text{and} \quad (\sqrt{3}-\sqrt{2})^{2n} = a_n - b_n\sqrt{6}$$

   (a) Determine the values of a_2 and b_2.
   (b) Prove that $2a_n - 1 < (\sqrt{3}+\sqrt{2})^{2n} < 2a_n$ for all positive integers n.
   (c) Let d_n be the ones (units) digit of the number $(\sqrt{3}+\sqrt{2})^{2n}$ when it is written in decimal form. Determine, with justification, the value of

$$d_1 + d_2 + d_3 + \cdots + d_{1865} + d_{1866} + d_{1867}$$

   (The given sum has 1867 terms.)