The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Wednesday, November 25, 2015
(in North America and South America)

Thursday, November 26, 2015
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 2 hours

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.**
   For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the Web site, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. If $\dfrac{8}{24} = \dfrac{4}{x+3}$, what is the value of x?

2. Let A, B and C be non-zero digits, so that BC is a two-digit positive integer and ABC is a three-digit positive integer made up of the digits A, B and C. Suppose that

$$\begin{array}{r} B\ C \\ A\ B\ C \\ +\ A\ B\ C \\ \hline 8\ 7\ 6 \end{array}$$

   What is the value of $A + B + C$?

3. A 5 m $\times$ 5 m flat square roof receives 6 mm of rainfall. All of this water (and no other water) drains into an empty rain barrel. The rain barrel is in the shape of a cylinder with a diameter of 0.5 m and a height of 1 m. Rounded to the nearest tenth of a percent, what percentage of the barrel will be full of water?

4. Determine all values of x for which $\left(2 \cdot 4^{x^2-3x}\right)^2 = 2^{x-1}$.

5. In a psychology experiment, an image of a cat or an image of a dog is flashed briefly onto a screen and then Anna is asked to guess whether the image showed a cat or a dog. This process is repeated a large number of times with an equal number of images of cats and images of dogs shown. If Anna is correct 95% of the time when she guesses "dog" and 90% of the time when she guesses "cat", determine the ratio of the number of times she *guessed* "dog" to the number of times she *guessed* "cat".

6. Suppose that X and Y are angles with $\tan X = \frac{1}{m}$ and $\tan Y = \frac{a}{n}$ for some positive integers a, m and n. Determine the number of positive integers $a \leq 50$ for which there are exactly 6 pairs of positive integers (m, n) with $X + Y = 45°$.

(Note: The formula $\tan(X+Y) = \dfrac{\tan X + \tan Y}{1 - \tan X \tan Y}$ may be useful.)

PART B

For each question in Part B, your solution must be well organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

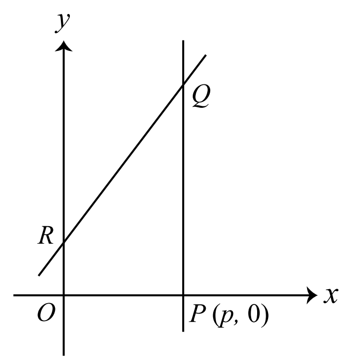
1. The line $y = 2x + 4$ intersects the y-axis at R, as shown. A second line, parallel to the y-axis, is drawn through $P(p, 0)$, with $p > 0$. These two lines intersect at Q.

   (a) Determine the length of OR. (Note that O is the origin $(0, 0)$.)

   (b) Determine the coordinates of point Q in terms of p.

   (c) If $p = 8$, determine the area of $OPQR$.

   (d) If the area of $OPQR$ is 77, determine the value of p.



2. (a) If $f(x) = \dfrac{x}{x-1}$ for $x \neq 1$, determine all real numbers $r \neq 1$ for which $f(r) = r$.

   (b) If $f(x) = \dfrac{x}{x-1}$ for $x \neq 1$, show that $f(f(x)) = x$ for all real numbers $x \neq 1$.

   (c) Suppose that k is a real number. Define $g(x) = \dfrac{2x}{x+k}$ for $x \neq -k$. Determine all real values of k for which $g(g(x)) = x$ for every real number x with $x \neq -k$ and $g(x) \neq -k$.

   (d) Suppose that a, b and c are non-zero real numbers. Define $h(x) = \dfrac{ax+b}{bx+c}$ for $x \neq -\dfrac{c}{b}$. Determine all triples (a, b, c) for which $h(h(x)) = x$ for every real number x with $x \neq -\dfrac{c}{b}$ and $h(x) \neq -\dfrac{c}{b}$.

3. Given a sequence $a_1, a_2, a_3, \ldots$ of positive integers, we define a new sequence $b_1, b_2, b_3, \ldots$ by $b_1 = a_1$ and, for every positive integer $n \geq 1$,

$$b_{n+1} = \begin{cases} b_n + a_{n+1} & \text{if } b_n \leq a_{n+1} \\ b_n - a_{n+1} & \text{if } b_n > a_{n+1} \end{cases}$$

For example, when $a_1, a_2, a_3, \cdots$ is the sequence $1, 2, 1, 2, 1, 2, \ldots$ we have

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
a_n	1	2	1	2	1	2	1	2	1	2	$\cdots$
b_n	1	3	2	4	3	1	2	4	3	1	$\cdots$

(a) Suppose that $a_n = n^2$ for all $n \geq 1$. Determine the value of b_{10}.

(b) Suppose that $a_n = n$ for all $n \geq 1$. Determine all positive integers n with $n < 2015$ for which $b_n = 1$.

(c) A sequence $x_1, x_2, x_3, \ldots$ is called *eventually periodic* if there exist positive integers r and p for which $x_{n+p} = x_n$ for all $n \geq r$.

Suppose that $a_1, a_2, a_3, \ldots$ is eventually periodic. Prove that $b_1, b_2, b_3, \ldots$ is eventually periodic.