The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2014 Fryer Contest

Wednesday, April 16, 2014
(in North America and South America)

Thursday, April 17, 2014
(outside of North America and South America)

Solutions

1. (a) There are 99 positive integers from 1 to 99.
The first 9 of these (the integers from 1 to 9) are each single-digit integers and thus contribute $1 \times 9 = 9$ digits to the given integer.
The final 90 positive integers (the integers from 10 to 99) are each 2-digit integers and thus contribute $2 \times 90 = 180$ digits to the given integer.
Therefore, the given integer has $9 + 180 = 189$ digits in total.

(b) From part (a), the integer formed by writing the positive integers from 1 to 99 in order next to each other has 189 digits.
In addition to these first 99 positive integers, the given integer has the positive integers from 100 to 199 written in order next to one another.
Since there are $199 - 100 + 1 = 100$ of these and each is a 3-digit integer, then they contribute $3 \times 100 = 300$ more digits.
Therefore, the given integer has $189 + 300 = 489$ digits in total.

(c) From part (b), the positive integers from 1 to 199 form a 489-digit integer.
To obtain 1155 digits, another $1155 - 489 = 666$ digits are required.
Assuming that each of the integers from 200 to n is a 3-digit integer (we will confirm this), then another $\frac{666}{3} = 222$ integers are needed to obtain 1155 digits in total.
(Since 222 integers beyond the integer 199 are needed to obtain 1155 digits in total, then each of these integers is indeed a 3-digit integer.)
Therefore, n is 222 integers past 199, or $n = 199 + 222$ and so $n = 421$.
If the positive integers from 1 to 421 are written in order next to each other, the resulting integer has 1155 digits.

(d) From part (c), the positive integers from 1 to 421 form a 1155 digit integer.
To obtain 1358 digits, another $1358 - 1155 = 203$ digits are required.
Writing the 68 integers that follow 421 will add another $3 \times 68 = 204$ digits to our total (since each of the 68 integers that follow 421 is a 3-digit integer).
The 68^{th} integer after 421 is $421 + 68 = 489$.
However, we only require another 203 digits, not 204. Since the 204^{th} digit is the final digit of 489, which is 9, then the 203^{rd} digit is 8.
Therefore, if the positive integers from 1 to 1000 are written in order next to each other, the 1358^{th} digit of the resulting integer is 8.

2. (a) The sum of the 3 angles in any triangle is 180°.
Therefore, $\angle BAC = 180^\circ - \angle ABC - \angle ACB = 180^\circ - 60^\circ - 50^\circ = 70^\circ$.

(b) Since BD bisects $\angle ABC$, then $\angle DBC = \frac{60^\circ}{2} = 30^\circ$.
Since CD bisects $\angle ACB$, then $\angle DCB = \frac{50^\circ}{2} = 25^\circ$.
Since the sum of the 3 angles in any triangle is 180°,

$$\angle BDC = 180^\circ - \angle DBC - \angle DCB = 180^\circ - 30^\circ - 25^\circ = 125^\circ.$$

(c) In $\triangle SQR$, let the measure of $\angle SQR$ be x°. Since $\triangle SQR$ is isosceles with $QS = RS$, then $\angle SRQ = \angle SQR = x^\circ$.
The sum of the 3 angles in $\triangle SQR$ is 180°, so $\angle SQR + \angle SRQ + \angle QSR = 180^\circ$, or $x^\circ + x^\circ + 140^\circ = 180^\circ$ or $2x = 40$, and so $x = 20$.
Since QS is the angle bisector of $\angle PQR$, then $\angle PQS = \angle SQR = x^\circ$.
Similarly, since RS is the angle bisector of $\angle PRQ$, then $\angle PRS = \angle SRQ = x^\circ$.
In $\triangle PQR$, $\angle PQR = \angle PQS + \angle SQR = (2x)^\circ$ and $\angle PRQ = \angle PRS + \angle SRQ = (2x)^\circ$.
The sum of the 3 angles in $\triangle PQR$ is 180°, so $\angle QPR = 180^\circ - \angle PQR - \angle PRQ$, or $\angle QPR = 180^\circ - (2x)^\circ - (2x)^\circ = 180^\circ - (4x)^\circ = 180^\circ - (4 \times 20)^\circ$ or $\angle QPR = 100^\circ$.

(d) We begin by assuming that it is possible that $\angle QSR = 80^\circ$.
Proceeding as we did in part (c), if we let the measure of $\angle SQR$ be y°, then $\angle SRQ = \angle SQR = y^\circ$.
In $\triangle SQR$, $\angle SQR + \angle SRQ + \angle QSR = 180^\circ$, or $y^\circ + y^\circ + 80^\circ = 180^\circ$ or $2y = 100$, and so $y = 50$.
In $\triangle PQR$, $\angle PQR = \angle PQS + \angle SQR = (2y)^\circ$ and $\angle PRQ = \angle PRS + \angle SRQ = (2y)^\circ$ (QS and RS are angle bisectors).
Thus, $\angle QPR = 180^\circ - \angle PQR - \angle PRQ$, or $\angle QPR = 180^\circ - (2y)^\circ - (2y)^\circ = 180^\circ - (4y)^\circ$, and so $\angle QPR = 180^\circ - (4 \times 50)^\circ$ or $\angle QPR = -20^\circ$.
Since all angles must have positive measure, then the only assumption that we made, (that $\angle QSR = 80^\circ$), must be false.
Therefore, it is not possible that $\angle QSR = 80^\circ$.

3. (a) Since the base BC of $\triangle ABC$ is horizontal, its length is given by the positive difference between the x-coordinates of points B and C, or $10 - 0 = 10$.
The height of $\triangle ABC$ is given by the length of the vertical line segment from point A to the base BC, which is the y-coordinate of point A, or 9 (since BC lies along the x-axis).
Thus, the area of $\triangle ABC$ before the first move in the game is made is $\frac{1}{2} \times 10 \times 9 = 45$.

(b) At this point in the game, the area of $\triangle ABC$ is $\frac{1}{2} \times 10 \times 7$ or 35 (since the length of the base is still 10 and the height is 7).
The person who makes the area of $\triangle ABC$ equal to 25 wins the game.
The base of $\triangle ABC$, BC, remains fixed at length 10 throughout the game since only point A can move.
The area of $\triangle ABC$ is 25 when the height of the triangle is 5, since $\frac{1}{2} \times 10 \times 5 = 25$.
The height of $\triangle ABC$ is 5 when the y-coordinate of point A is 5.
That is, the only way for a player to win the game is to make the move that changes the y-coordinate of point A to 5.

It is now Dexter's turn to move. Dexter can move A down one unit to the point $(2, 6)$ or he may move A left one unit to the point $(1, 7)$.
If Dexter moves A down one unit to the point $(2, 6)$, then on the next turn Ella can move A down one more unit to the point $(2, 5)$ and win the game (since the y-coordinate would be 5).
Alternatively, if Dexter moves A one unit left to the point $(1, 7)$, then on the next turn Ella can move A one more unit left to the point $(0, 7)$ (Ella would not choose to move A down one unit to $(1, 6)$ since then Dexter could win on his next move).
On the next turn, Dexter is forced to move A down one unit to $(0, 6)$ since moving A left would make the x-coordinate negative which is not permitted.
On the next move, Ella can move A down one more unit to $(0, 5)$ and win the game.
We have shown that for all possible moves that Dexter can make, Ella can always win the game from the point $(2, 7)$.

(c) (i) Beginning with point A at $(6, 9)$, the player who moves second, Geoff, has the winning strategy.
This winning strategy will be first described and then justified in parts (ii) and (iii) below.

(ii) There are many different ways to describe Geoff's winning strategy.
One way to describe it is to say that on Geoff's turn, he will perform the same move (that is, move A either left or down) that his opponent Faisal just performed in the previous move.

(iii) Why does Geoff win the game every time if he always matches the move that Faisal just performed?
Begin by assuming that Geoff can always match Faisal's move (an assumption that we will prove later on).
At the start of the game, the y-coordinate of point A is 9.
If Faisal moves down to 8, then Geoff will match and move down to 7.
If Faisal moves down to 6, then Geoff will match and move down to 5.
From part (b), we know that the player who makes the y-coordinate of point A equal to 5 wins the game.
Thus Geoff wins the game, assuming that Geoff can always move down after Faisal and is never forced to move down before Faisal.

Next, we must prove our assumption that Geoff can always match Faisal's move.
First, consider moves down.
A move down followed by a matching move down is always possible since the game will end when the the y-coordinate of A is 5 (and as was already shown, it is Geoff who makes this final move).
That is, the y-coordinate of A will never be made negative and Geoff can always match any down move that Faisal makes.
Finally, we consider moves left.
At the start of the game, the x-coordinate of point A is 6, an even integer.
If at any time in the game Faisal moves A left one unit, then the x-coordinate becomes 5, an odd integer.
On his next turn, Geoff matches Faisal's move (left), and the x-coordinate becomes 4, an even integer again.
That is, each time Faisal moves A to the left, the x-coordinate becomes an odd integer, and since Geoff matches Faisal's move he returns the x-coordinate to an even integer.
Since the smallest x-coordinate that is permitted is an even integer, 0 (coordinates cannot be made negative), it is Geoff who will (if required) make the last possible move to the left.
On his next turn, Faisal is then forced to move A downward.

Therefore we have shown that Geoff will always be able to match Faisal's move, and that as a result, he will be the player who moves point A so that the y-coordinate becomes 5, thus winning the game.

4. (a) The eight subsets of the set $\{1, 2, 3\}$ are: $\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$, and $\{1, 2, 3\}$.
Therefore the eight subset sums are: $0, 1, 2, 3, 3, 4, 5$, and 6.

(b) When constructing subsets, any element of the original set may either be included in the subset or not included in the subset.
That is, when constructing a subset from the set $\{1, 2, 3, 4, 5\}$, there are two choices for each of the 5 elements of the set; include it in the subset or do not.
Since there are two choices for each of the 5 elements, there are a total of 2^5 or 32 different subsets that can be created.
Consider the number of these 32 subsets which contain the number 1.
That is, when counting the number of these subsets, there is only one choice for the number one (include it in the subset), while there are still two choices for each of the numbers $2, 3, 4$, and 5.
Thus, there are 2^4 or 16 subsets that contain the number 1 (and so there are $32 - 16 = 16$ subsets which do not contain the number 1).

This is true in general for each of the elements of the original set; the number 2 will be included in 16 of the 32 subsets, the number 3 will be included in 16 subsets, and likewise for the numbers 4 and 5.
Since the element 1 is contained in exactly 16 subsets, then these 16 occurrences of 1 contribute exactly $1 \times 16 = 16$ to the sum of the subsets that they appear in.
Similarly, since the element 2 is contained in exactly 16 subsets, then these 16 occurrences of 2 contribute exactly $2 \times 16 = 32$ to the sum of the subsets that they appear in.
Since each of the 5 elements is contained in exactly 16 subsets, then the total contribution of all 5 elements in all possible subsets is

$$(1 \times 16) + (2 \times 16) + (3 \times 16) + (4 \times 16) + (5 \times 16) = 16(1 + 2 + 3 + 4 + 5) = 16(15) = 240.$$

Thus, the sum of all of the subset sums of $\{1, 2, 3, 4, 5\}$ is 240.

(c) Each subset of $\{1, 3, 4, 5, 7, 8, 12, 16\}$ consists of zero or more of the odd integers $\{1, 3, 5, 7\}$ and zero or more of the multiples of four $\{4, 8, 12, 16\}$.
If a subset is formed using only numbers that are divisible by 4, then the subset sum must also be divisible by 4 (since the sum of multiples of 4 is a multiple of 4).
So then every subset of $\{4, 8, 12, 16\}$ must have a subset sum that is divisible by 4.
As in part (b), each of the 4 elements will appear in $2^3 = 8$ of the $2^4 = 16$ subsets. The sum of all subset sums using only elements from the set $\{4, 8, 12, 16\}$ is then
$(4 \times 8) + (8 \times 8) + (12 \times 8) + (16 \times 8) = 8(4 + 8 + 12 + 16) = 320$.

Any subset includes $0, 1, 2, 3$ or 4 of the elements of $\{4, 8, 12, 16\}$.
The sum of these elements is divisible by 4.
For the sum of all elements in the subset to be divisible by 4, the sum of the remaining (odd) elements must be divisible by 4.
Some combinations of the remaining (odd) elements $\{1, 3, 5, 7\}$ produce sums which are divisible by 4.
Specifically, these are $\{1, 3\}$, $\{1, 7\}$, $\{3, 5\}$, $\{5, 7\}$, $\{1, 3, 5, 7\}$, and the empty set $\{\}$.
(No other subsets of $\{1, 3, 5, 7\}$ have a sum divisible by 4 since each of the individual elements is odd, thus the sum of any 3 of them is odd, and the only pairs remaining are $\{1, 5\}$ and $\{3, 7\}$, whose sums are 6 and 10).
Since each of these subsets has a sum which is divisible by 4, then combining them (that is, using all of their elements) with any of the 16 subsets of $\{4, 8, 12, 16\}$ (including the empty set) will produce a new subset whose sum is also divisible by 4.
What remains is to determine the sum of all of these subset sums.
First consider combining both of the elements $\{1, 3\}$ with each of the 16 subsets of $\{4, 8, 12, 16\}$. As we previously determined, the sum of all of the subsets of $\{4, 8, 12, 16\}$ is 320.
Combining both of the elements $\{1, 3\}$ with each of these subsets adds $1 + 3 = 4$ to each of the subset sums.
Since there are 16 subsets, then the sum of all subset sums of this form is $(4 \times 16) + 320$ or 384.
Similarly, combining both of the elements $\{1, 7\}$ with each of the subsets of $\{4, 8, 12, 16\}$ adds $1 + 7 = 8$ to each of the subset sums.
Since there are 16 subsets, then the sum of all subset sums of this form is $(8 \times 16) + 320$ or 448.
The sum of all subset sums which combine both $\{3, 5\}$ with all subsets of $\{4, 8, 12, 16\}$ is $(8 \times 16) + 320$ or 448.
The sum of all subset sums which combine both $\{5, 7\}$ with all subsets of $\{4, 8, 12, 16\}$ is

$(12 \times 16) + 320$ or 512.
The sum of all subset sums which combine all of $\{1, 3, 5, 7\}$ with all subsets of $\{4, 8, 12, 16\}$ is $(16 \times 16) + 320$ or 576.
Finally, combining the empty set with each of the subsets of $\{4, 8, 12, 16\}$ adds nothing to each of the subset sums and so the sum of all subset sums is 320 ($(0 \times 16) + 320 = 320$).
Since there are no other combinations of elements which produce subset sums that are divisible by 4, the sum of all required subset sums is $384+448+448+512+576+320 = 2688$.