The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2014 Euclid Contest

Tuesday, April 15, 2014
(in North America and South America)

Wednesday, April 16, 2014
(outside of North America and South America)

Solutions

1. (a) Evaluating, $\dfrac{\sqrt{16}+\sqrt{9}}{\sqrt{16+9}} = \dfrac{4+3}{\sqrt{25}} = \dfrac{7}{5}$.

   (b) Since the sum of the angles in a triangle is 180°, then $(x-10)^\circ + (x+10)^\circ + x^\circ = 180^\circ$ or $(x-10)+(x+10)+x = 180$.
   Thus, $3x = 180$ and so $x = 60$.

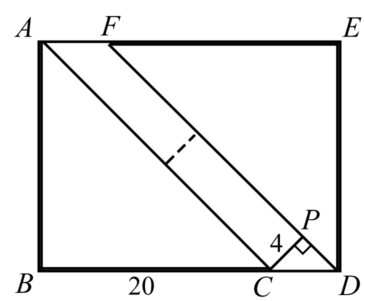
   (c) Suppose Bart earns $\$x$ per hour. In 4 hours, he earns $4 \times \$x = \$4x$.
   Then Lisa earns $\$2x$ per hour. In 6 hours, she earns $6 \times \$2x = \$12x$.
   Since they earn \$200 in total, then $4x + 12x = 200$ or $16x = 200$.
   Therefore, $x = 12.5$.
   Finally, since $2x = 25$, then Lisa earns \$25 per hour.

2. (a) The perimeter of the region includes the diameter and the semi-circle.
   Since the radius of the region is 10, then the length of its diameter is 20.
   Since the radius of the region is 10, then the circumference of an entire circle with this radius is $2\pi(10) = 20\pi$, so the arc length of the semi-circle is one-half of 20π, or 10π.
   Therefore, the perimeter of the region is $10\pi + 20$.

   (b) The x-intercepts of the parabola with equation $y = 10(x+2)(x-5)$ are -2 and 5.
   Since the line segment, PQ, joining these points is horizontal, then its length is the difference in the intercepts, or $5-(-2) = 7$.

   (c) The slope of the line joining the points $C(0,60)$ and $D(30,0)$ is $\dfrac{60-0}{0-30} = \dfrac{60}{-30} = -2$.
   Since this line passes through $C(0,60)$, then the y-intercept of the line is 60, and so an equation of the line is $y = -2x + 60$.
   We thus want to find the point of intersection, E, between the lines with equations $y = -2x + 60$ and $y = 2x$.
   Equating y-coordinates, we obtain $-2x + 60 = 2x$ or $4x = 60$, and so $x = 15$.
   Substituting $x = 15$ into the equation $y = 2x$, we obtain $y = 2(15) = 30$.
   Therefore, the coordinates of E are $(15, 30)$.

3. (a) We note that $BD = BC + CD$ and that $BC = 20$ cm, so we need to determine CD.



   We draw a line from C to P on FD so that CP is perpendicular to DF.
   Since AC and DF are parallel, then CP is also perpendicular to AC.
   The distance between AC and DF is 4 cm, so $CP = 4$ cm.
   Since $\triangle ABC$ is isosceles and right-angled, then $\angle ACB = 45^\circ$.
   Thus, $\angle PCD = 180^\circ - \angle ACB - \angle PCA = 180^\circ - 45^\circ - 90^\circ = 45^\circ$.
   Since $\triangle CPD$ is right-angled at P and $\angle PCD = 45^\circ$, then $\triangle CPD$ is also an isosceles right-angled triangle.
   Therefore, $CD = \sqrt{2}CP = 4\sqrt{2}$ cm.
   Finally, $BD = BC + CD = (20 + 4\sqrt{2})$ cm.

(b) Manipulating the given equation and noting that $x \neq 0$ and $x \neq -\frac{1}{2}$ since neither denominator can equal 0, we obtain

$$\begin{aligned}
\frac{x^2+x+4}{2x+1} &= \frac{4}{x} \\
x(x^2+x+4) &= 4(2x+1) \\
x^3+x^2+4x &= 8x+4 \\
x^3+x^2-4x-4 &= 0 \\
x^2(x+1)-4(x+1) &= 0 \\
(x+1)(x^2-4) &= 0 \\
(x+1)(x-2)(x+2) &= 0
\end{aligned}$$

Therefore, $x=-1$ or $x=2$ or $x=-2$. We can check by substitution that each satisfies the original equation.

4. (a) *Solution 1*

Since $900 = 30^2$ and $30 = 2 \times 3 \times 5$, then $900 = 2^2 3^2 5^2$.
The positive divisors of 900 are those integers of the form $d = 2^a 3^b 5^c$, where each of a, b, c is 0, 1 or 2.
For d to be a perfect square, the exponent on each prime factor in the prime factorization of d must be even.
Thus, for d to be a perfect square, each of a, b, c must be 0 or 2.
There are two possibilities for each of a, b, c so $2 \times 2 \times 2 = 8$ possibilities for d.
These are $2^0 3^0 5^0 = 1$, $2^2 3^0 5^0 = 4$, $2^0 3^2 5^0 = 9$, $2^0 3^0 5^2 = 25$, $2^2 3^2 5^0 = 36$, $2^2 3^0 5^2 = 100$, $2^0 3^2 5^2 = 225$, and $2^2 3^2 5^2 = 900$.
Thus, 8 of the positive divisors of 900 are perfect squares.

Solution 2

The positive divisors of 900 are

$$1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 25, 30, 36, 45, 50, 60, 75, 90, 100, 150, 180, 225, 300, 450, 900$$

Of these, 1, 4, 9, 25, 36, 100, 225, and 900 are perfect squares ($1^2, 2^2, 3^2, 5^2, 6^2, 10^2, 15^2, 30^2$, respectively).
Thus, 8 of the positive divisors of 900 are perfect squares.

(b) In isosceles triangle ABC, $\angle ABC = \angle ACB$, so the sides opposite these angles (AC and AB, respectively) are equal in length.
Since the vertices of the triangle are $A(k,3)$, $B(3,1)$ and $C(6,k)$, then we obtain

$$\begin{aligned}
AC &= AB \\
\sqrt{(k-6)^2+(3-k)^2} &= \sqrt{(k-3)^2+(3-1)^2} \\
(k-6)^2+(3-k)^2 &= (k-3)^2+(3-1)^2 \\
(k-6)^2+(k-3)^2 &= (k-3)^2+2^2 \\
(k-6)^2 &= 4
\end{aligned}$$

Thus, $k-6=2$ or $k-6=-2$, and so $k=8$ or $k=4$.
We can check by substitution that each satisfies the original equation.

5. (a) Bottle A contains 40 g of which 10% is acid.
Thus, it contains $0.1 \times 40 = 4$ g of acid and $40 - 4 = 36$ g of water.
Bottle B contains 50 g of which 20% is acid.
Thus, it contains $0.2 \times 50 = 10$ g of acid and $50 - 10 = 40$ g of water.
Bottle C contains 50 g of which 30% is acid.
Thus, it contains $0.3 \times 50 = 15$ g of acid and $50 - 15 = 35$ g of water.
In total, the three bottles contain $40 + 50 + 50 = 140$ g, of which $4 + 10 + 15 = 29$ g is acid and $140 - 29 = 111$ g is water.
The new mixture has mass 60 g of which 25% is acid.
Thus, it contains $0.25 \times 60 = 15$ g of acid and $60 - 15 = 45$ g of water.
Since the total mass in the three bottles is initially 140 g and the new mixture has mass 60 g, then the remaining contents have mass $140 - 60 = 80$ g.
Since the total mass of acid in the three bottles is initially 29 g and the acid in the new mixture has mass 15 g, then the acid in the remaining contents has mass $29 - 15 = 14$ g.
This remaining mixture is thus $\dfrac{14\text{ g}}{80\text{ g}} \times 100\% = 17.5\%$ acid.

(b) Since $3x + 4y = 10$, then $4y = 10 - 3x$.
Therefore, when $3x + 4y = 10$,

$$\begin{aligned} x^2 + 16y^2 &= x^2 + (4y)^2 \\ &= x^2 + (10 - 3x)^2 \\ &= x^2 + (9x^2 - 60x + 100) \\ &= 10x^2 - 60x + 100 \\ &= 10(x^2 - 6x + 10) \\ &= 10(x^2 - 6x + 9 + 1) \\ &= 10((x-3)^2 + 1) \\ &= 10(x-3)^2 + 10 \end{aligned}$$

Since $(x-3)^2 \geq 0$, then the minimum possible value of $10(x-3)^2 + 10$ is $10(0) + 10 = 10$.
This occurs when $(x-3)^2 = 0$ or $x = 3$.
Therefore, the minimum possible value of $x^2 + 16y^2$ when $3x + 4y = 10$ is 10.

6. (a) *Solution 1*
Suppose that the bag contains g gold balls.
We assume that Feridun reaches into the bag and removes the two balls one after the other.
There are 40 possible balls that he could remove first and then 39 balls that he could remove second. In total, there are 40(39) pairs of balls that he could choose in this way.
If he removes 2 gold balls, then there are g possible balls that he could remove first and then $g - 1$ balls that he could remove second. In total, there are $g(g-1)$ pairs of gold balls that he could remove.
We are told that the probability of removing 2 gold balls is $\dfrac{5}{12}$.
Since there are 40(39) total pairs of balls that can be chosen and $g(g-1)$ pairs of gold balls that can be chosen in this way, then $\dfrac{g(g-1)}{40(39)} = \dfrac{5}{12}$ which is equivalent to $g(g-1) = \dfrac{5}{12}(40)(39) = 650$.

Therefore, $g^2 - g - 650 = 0$ or $(g-26)(g+25) = 0$, and so $g = 26$ or $g = -25$.
Since $g > 0$, then $g = 26$, so there are 26 gold balls in the bag.

Solution 2
Suppose that the bag contains g gold balls.
We assume that Feridun reaches into the bag and removes the two balls together.
Since there are 40 balls in the bag, there are $\binom{40}{2}$ pairs of balls that he could choose in this way.
Since there are g gold balls in the bag, then there are $\binom{g}{2}$ pairs of gold balls that he could choose in this way.
We are told that the probability of removing 2 gold balls is $\frac{5}{12}$.
Since there are $\binom{40}{2}$ pairs in total that can be chosen and $\binom{g}{2}$ pairs of gold balls that can be chosen in this way, then $\dfrac{\binom{g}{2}}{\binom{40}{2}} = \dfrac{5}{12}$ which is equivalent to $\binom{g}{2} = \dfrac{5}{12}\binom{40}{2}$.
Since $\binom{n}{2} = \dfrac{n(n-1)}{2}$, then this equation is equivalent to $\dfrac{g(g-1)}{2} = \dfrac{5}{12}\dfrac{40(39)}{2} = 325$.
Therefore, $g(g-1) = 650$ or $g^2 - g - 650 = 0$ or $(g-26)(g+25) = 0$, and so $g = 26$ or $g = -25$.
Since $g > 0$, then $g = 26$, so there are 26 gold balls in the bag.

(b) Suppose that the first term in the geometric sequence is $t_1 = a$ and the common ratio in the sequence is r.
Then the sequence, which has n terms, is $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$.
In general, the kth term is $t_k = ar^{k-1}$; in particular, the nth term is $t_n = ar^{n-1}$.
Since $t_1 t_n = 3$, then $a \cdot ar^{n-1} = 3$ or $a^2 r^{n-1} = 3$.
Since $t_1 t_2 \cdots t_{n-1} t_n = 59\,049$, then

$$\begin{aligned}
(a)(ar)\cdots(ar^{n-2})(ar^{n-1}) &= 59\,049 \\
a^n r r^2 \cdots r^{n-2} r^{n-1} &= 59\,049 \qquad \text{(since there are } n \text{ factors of } a \text{ on the left side)} \\
a^n r^{1+2+\cdots+(n-2)+(n-1)} &= 59\,049 \\
a^n r^{\frac{1}{2}(n-1)(n)} &= 59\,049
\end{aligned}$$

since $1 + 2 + \cdots + (n-2) + (n-1) = \frac{1}{2}(n-1)(n)$.
Since $a^2 r^{n-1} = 3$, then $(a^2 r^{n-1})^n = 3^n$ or $a^{2n} r^{(n-1)(n)} = 3^n$.
Since $a^n r^{\frac{1}{2}(n-1)(n)} = 59\,049$, then $\left(a^n r^{\frac{1}{2}(n-1)(n)}\right)^2 = 59\,049^2$ or $a^{2n} r^{(n-1)(n)} = 59\,049^2$.
Since the left sides of these equations are the same, then $3^n = 59\,049^2$.
Now

$$59\,049 = 3(19\,683) = 3^2(6561) = 3^3(2187) = 3^4(729) = 3^5(243) = 3^6(81) = 3^6 3^4 = 3^{10}$$

Since $59\,049 = 3^{10}$, then $59\,049^2 = 3^{20}$ and so $3^n = 3^{20}$, which gives $n = 20$.

7. (a) Let $a = x - 2013$ and let $b = y - 2014$.
The given equation becomes $\dfrac{ab}{a^2+b^2} = -\dfrac{1}{2}$, which is equivalent to $2ab = -a^2 - b^2$ and $a^2 + 2ab + b^2 = 0$.
This is equivalent to $(a+b)^2 = 0$ which is equivalent to $a + b = 0$.
Since $a = x - 2013$ and $b = y - 2014$, then $x - 2013 + y - 2014 = 0$ or $x + y = 4027$.

(b) Let $a = \log_{10} x$.
Then $(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10\,000$ becomes $a^{\log_{10} a} = 10^4$.
Taking the base 10 logarithm of both sides and using the fact that $\log_{10}(a^b) = b\log_{10} a$, we obtain $(\log_{10} a)(\log_{10} a) = 4$ or $(\log_{10} a)^2 = 4$.
Therefore, $\log_{10} a = \pm 2$ and so $\log_{10}(\log_{10} x) = \pm 2$.

If $\log_{10}(\log_{10} x) = 2$, then $\log_{10} x = 10^2 = 100$ and so $x = 10^{100}$.
If $\log_{10}(\log_{10} x) = -2$, then $\log_{10} x = 10^{-2} = \frac{1}{100}$ and so $x = 10^{1/100}$.
Therefore, $x = 10^{100}$ or $x = 10^{1/100}$.

We check these answers in the original equation.
If $x = 10^{100}$, then $\log_{10} x = 100$.
Thus, $(\log_{10} x)^{\log_{10}(\log_{10} x)} = 100^{\log_{10} 100} = 100^2 = 10\,000$.
If $x = 10^{1/100}$, then $\log_{10} x = 1/100 = 10^{-2}$.
Thus, $(\log_{10} x)^{\log_{10}(\log_{10} x)} = (10^{-2})^{\log_{10}(10^{-2})} = (10^{-2})^{-2} = 10^4 = 10\,000$.

8. (a) We use the cosine law in $\triangle ABD$ to determine the length of BD:

$$BD^2 = AB^2 + AD^2 - 2(AB)(AD)\cos(\angle BAD)$$

We are given that $AB = 75$ and $AD = 20$, so we need to determine $\cos(\angle BAD)$.
Now

$$\begin{aligned}\cos(\angle BAD) &= \cos(\angle BAC + \angle EAD)\\ &= \cos(\angle BAC)\cos(\angle EAD) - \sin(\angle BAC)\sin(\angle EAD)\\ &= \frac{AC}{AB}\frac{AD}{AE} - \frac{BC}{AB}\frac{ED}{AE}\end{aligned}$$

since $\triangle ABC$ and $\triangle ADE$ are right-angled.
Since $AB = 75$ and $BC = 21$, then by the Pythagorean Theorem,

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{75^2 - 21^2} = \sqrt{5625 - 441} = \sqrt{5184} = 72$$

since $AC > 0$.
Since $AC = 72$ and $CE = 47$, then $AE = AC - CE = 25$.
Since $AE = 25$ and $AD = 20$, then by the Pythagorean Theorem,

$$ED = \sqrt{AE^2 - AD^2} = \sqrt{25^2 - 20^2} = \sqrt{625 - 400} = \sqrt{225} = 15$$

since $ED > 0$.
Therefore,

$$\cos(\angle BAD) = \frac{AC}{AB}\frac{AD}{AE} - \frac{BC}{AB}\frac{ED}{AE} = \frac{72}{75}\frac{20}{25} - \frac{21}{75}\frac{15}{25} = \frac{1440 - 315}{75(25)} = \frac{1125}{75(25)} = \frac{45}{75} = \frac{3}{5}$$

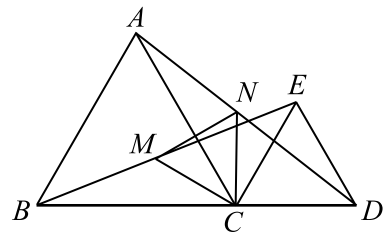
Finally,

$$\begin{aligned} BD^2 &= AB^2 + AD^2 - 2(AB)(AD)\cos(\angle BAD) \\ &= 75^2 + 20^2 - 2(75)(20)(\tfrac{3}{5}) \\ &= 5625 + 400 - 1800 \\ &= 4225 \end{aligned}$$

Since $BD > 0$, then $BD = \sqrt{4225} = 65$, as required.

(b) *Solution 1*

Consider $\triangle BCE$ and $\triangle ACD$.



Since $\triangle ABC$ is equilateral, then $BC = AC$.
Since $\triangle ECD$ is equilateral, then $CE = CD$.
Since BCD is a straight line and $\angle ECD = 60^\circ$, then $\angle BCE = 180^\circ - \angle ECD = 120^\circ$.
Since BCD is a straight line and $\angle BCA = 60^\circ$, then $\angle ACD = 180^\circ - \angle BCA = 120^\circ$.
Therefore, $\triangle BCE$ is congruent to $\triangle ACD$ ("side-angle-side").
Since $\triangle BCE$ and $\triangle ACD$ are congruent and CM and CN are line segments drawn from the corresponding vertex (C in both triangles) to the midpoint of the opposite side, then $CM = CN$.
Since $\angle ECD = 60^\circ$, then $\triangle ACD$ can be obtained by rotating $\triangle BCE$ through an angle of 60° clockwise about C.
This means that after this 60° rotation, CM coincides with CN.
In other words, $\angle MCN = 60^\circ$.
But since $CM = CN$ and $\angle MCN = 60^\circ$, then

$$\angle CMN = \angle CNM = \tfrac{1}{2}(180^\circ - \angle MCN) = 60^\circ$$

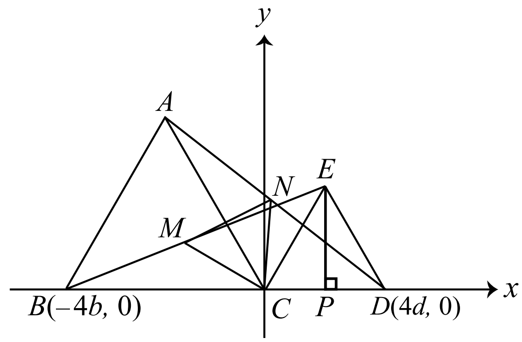
Therefore, $\triangle MNC$ is equilateral, as required.

Solution 2

We prove that $\triangle MNC$ is equilateral by introducing a coordinate system.

Suppose that C is at the origin $(0,0)$ with BCD along the x-axis, with B having coordinates $(-4b,0)$ and D having coordinates $(4d,0)$ for some real numbers $b,d>0$.

Drop a perpendicular from E to P on CD.



Since $\triangle ECD$ is equilateral, then P is the midpoint of CD.

Since C has coordinates $(0,0)$ and D has coordinates $(4d,0)$, then the coordinates of P are $(2d,0)$.

Since $\triangle ECD$ is equilateral, then $\angle ECD = 60^\circ$ and so $\triangle EPC$ is a 30°-60°-90° triangle and so $EP = \sqrt{3}CP = 2\sqrt{3}d$.

Therefore, the coordinates of E are $(2d, 2\sqrt{3}d)$.

In a similar way, we can show that the coordinates of A are $(-2b, 2\sqrt{3}b)$.

Now M is the midpoint of $B(-4b,0)$ and $E(2d,2\sqrt{3}d)$, so the coordinates of M are $\left(\frac{1}{2}(-4b+2d), \frac{1}{2}(0+2\sqrt{3}d)\right)$ or $(-2b+d, \sqrt{3}d)$.

Also, N is the midpoint of $A(-2b,2\sqrt{3}b)$ and $D(4d,0)$, so the coordinates of N are $\left(\frac{1}{2}(-2b+4d), \frac{1}{2}(2\sqrt{3}b+0)\right)$ or $(-b+2d, \sqrt{3}b)$.

To show that $\triangle MNC$ is equilateral, we show that $CM = CN = MN$ or equivalently that $CM^2 = CN^2 = MN^2$:

$$\begin{aligned}
CM^2 &= (-2b+d-0)^2 + (\sqrt{3}d-0)^2 \\
&= (-2b+d)^2 + (\sqrt{3}d)^2 \\
&= 4b^2 - 4bd + d^2 + 3d^2 \\
&= 4b^2 - 4bd + 4d^2
\end{aligned}$$

$$\begin{aligned}
CN^2 &= (-b+2d-0)^2 + (\sqrt{3}b-0)^2 \\
&= (-b+2d)^2 + (\sqrt{3}b)^2 \\
&= b^2 - 4bd + 4d^2 + 3b^2 \\
&= 4b^2 - 4bd + 4d^2
\end{aligned}$$

$$\begin{aligned}
MN^2 &= ((-2b+d)-(-b+2d))^2 + (\sqrt{3}d-\sqrt{3}b)^2 \\
&= (-b-d)^2 + 3(d-b)^2 \\
&= b^2 + 2bd + d^2 + 3d^2 - 6bd + 3b^2 \\
&= 4b^2 - 4bd + 4d^2
\end{aligned}$$

Therefore, $CM^2 = CN^2 = MN^2$ and so $\triangle MNC$ is equilateral, as required.

9. (a) Let $S = \sin^6 1^\circ + \sin^6 2^\circ + \sin^6 3^\circ + \cdots + \sin^6 87^\circ + \sin^6 88^\circ + \sin^6 89^\circ$.
Since $\sin\theta = \cos(90^\circ - \theta)$, then $\sin^6\theta = \cos^6(90^\circ - \theta)$, and so

$$\begin{aligned} S &= \sin^6 1^\circ + \sin^6 2^\circ + \cdots + \sin^6 44^\circ + \sin^6 45^\circ \\ &\qquad + \cos^6(90^\circ - 46^\circ) + \cos^6(90^\circ - 47^\circ) + \cdots + \cos^6(90^\circ - 89^\circ) \\ &= \sin^6 1^\circ + \sin^6 2^\circ + \cdots + \sin^6 44^\circ + \sin^6 45^\circ + \cos^6 44^\circ + \cos^6 43^\circ + \cdots + \cos^6 1^\circ \\ &= (\sin^6 1^\circ + \cos^6 1^\circ) + (\sin^6 2^\circ + \cos^6 2^\circ) + \cdots + (\sin^6 44^\circ + \cos^6 44^\circ) + \sin^6 45^\circ \end{aligned}$$

Since $\sin 45^\circ = \frac{1}{\sqrt{2}}$, then $\sin^6 45^\circ = \frac{1}{2^3} = \frac{1}{8}$.
Also, since

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = (x+y)((x+y)^2 - 3xy)$$

then substituting $x = \sin^2\theta$ and $y = \cos^2\theta$, we obtain

$$\begin{aligned} x^3 + y^3 &= (x+y)((x+y)^2 - 3xy) \\ \sin^6\theta + \cos^6\theta &= (\sin^2\theta + \cos^2\theta)((\sin^2\theta + \cos^2\theta)^2 - 3\sin^2\theta\cos^2\theta) \\ \sin^6\theta + \cos^6\theta &= 1(1 - 3\sin^2\theta\cos^2\theta) \end{aligned}$$

since $\sin^2\theta + \cos^2\theta = 1$.
Therefore,

$$\begin{aligned} S &= (\sin^6 1^\circ + \cos^6 1^\circ) + (\sin^6 2^\circ + \cos^6 2^\circ) + \cdots + (\sin^6 44^\circ + \cos^6 44^\circ) + \sin^6 45^\circ \\ &= (1 - 3\sin^2 1^\circ\cos^2 1^\circ) + (1 - 3\sin^2 2^\circ\cos^2 2^\circ) + \cdots + (1 - 3\sin^2 44^\circ\cos^2 44^\circ) + \tfrac{1}{8} \\ &= 44 - (3\sin^2 1^\circ\cos^2 1^\circ + 3\sin^2 2^\circ\cos^2 2^\circ + \cdots + 3\sin^2 44^\circ\cos^2 44^\circ) + \tfrac{1}{8} \\ &= \tfrac{353}{8} - \tfrac{3}{4}(4\sin^2 1^\circ\cos^2 1^\circ + 4\sin^2 2^\circ\cos^2 2^\circ + \cdots + 4\sin^2 44^\circ\cos^2 44^\circ) \end{aligned}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$, then $4\sin^2\theta\cos^2\theta = \sin^2 2\theta$, which gives

$$\begin{aligned} S &= \tfrac{353}{8} - \tfrac{3}{4}(4\sin^2 1^\circ\cos^2 1^\circ + 4\sin^2 2^\circ\cos^2 2^\circ + \cdots + 4\sin^2 44^\circ\cos^2 44^\circ) \\ &= \tfrac{353}{8} - \tfrac{3}{4}(\sin^2 2^\circ + \sin^2 4^\circ + \cdots + \sin^2 88^\circ) \\ &= \tfrac{353}{8} - \tfrac{3}{4}(\sin^2 2^\circ + \sin^2 4^\circ + \cdots + \sin^2 44^\circ + \sin^2 46^\circ + \cdots + \sin^2 86^\circ + \sin^2 88^\circ) \\ &= \tfrac{353}{8} - \tfrac{3}{4}(\sin^2 2^\circ + \sin^2 4^\circ + \cdots + \sin^2 44^\circ + \\ &\qquad \cos^2(90^\circ - 46^\circ) + \cdots + \cos^2(90^\circ - 86^\circ) + \cos^2(90^\circ - 88^\circ)) \\ &= \tfrac{353}{8} - \tfrac{3}{4}(\sin^2 2^\circ + \sin^2 4^\circ + \cdots + \sin^2 44^\circ + \cos^2 44^\circ + \cdots + \cos^2 4^\circ + \cos^2 2^\circ) \\ &= \tfrac{353}{8} - \tfrac{3}{4}((\sin^2 2^\circ + \cos^2 2^\circ) + (\sin^2 4^\circ + \cos^2 4^\circ) + \cdots + (\sin^2 44^\circ + \cos^2 44^\circ)) \\ &= \tfrac{353}{8} - \tfrac{3}{4}(22) \qquad (\text{since } \sin^2\theta + \cos^2\theta = 1) \\ &= \tfrac{353}{8} - \tfrac{132}{8} \\ &= \tfrac{221}{8} \end{aligned}$$

Therefore, since $S = \dfrac{m}{n}$, then $m = 221$ and $n = 8$ satisfy the required equation.

(b) First, we prove that $f(n) = \dfrac{n(n+1)(n+2)(n+3)}{24}$ in two different ways.

Method 1

If an n-digit integer has digits with a sum of 5, then there are several possibilities for the combination of non-zero digits used:

$$5 \quad 4,1 \quad 3,2 \quad 3,1,1 \quad 2,2,1 \quad 2,1,1,1 \quad 1,1,1,1,1$$

We count the number of possible integers in each case by determining the number of arrangements of the non-zero digits; we call the number of ways of doing this a. (For example, the digits 4 and 1 can be arranged as 4 1 or 1 4.) We then place the leftmost digit in such an arrangement as the leftmost digit of the n-digit integer (which must be non-zero) and choose the positions for the remaining non-zero digits among the remaining $n-1$ positions; we call the number of ways of doing this b. (For example, for the arrangement 1 4, the digit 1 is in the leftmost position and the digit 4 can be in any of the remaining $n-1$ positions.) We fill the rest of the positions with 0s. The number of possible integers in each case will be ab, since this method will create all such integers and for each of the a arrangements of the non-zero digits, there will be b ways of arranging the digits after the first one. We make a chart to summarize the cases, expanding each total and writing it as a fraction with denominator 24:

Case	a	b	ab (expanded)
5	1	1	$1 = \dfrac{24}{24}$
4, 1	2	$(n-1)$	$2(n-1) = \dfrac{48n-48}{24}$
3, 2	2	$(n-1)$	$2(n-1) = \dfrac{48n-48}{24}$
3, 1, 1	3	$\dbinom{n-1}{2}$	$3\dbinom{n-1}{2} = \dfrac{36n^2-108n+72}{24}$
2, 2, 1	3	$\dbinom{n-1}{2}$	$3\dbinom{n-1}{2} = \dfrac{36n^2-108n+72}{24}$
2, 1, 1, 1	4	$\dbinom{n-1}{3}$	$4\dbinom{n-1}{3} = \dfrac{16n^3-96n^2+176n-96}{24}$
1, 1, 1, 1, 1	1	$\dbinom{n-1}{4}$	$\dbinom{n-1}{4} = \dfrac{n^4-10n^3+35n^2-50n+24}{24}$

(Note that in the second and third cases we need $n \geq 2$, in the fourth and fifth cases we need $n \geq 3$, in the sixth case we need $n \geq 4$, and the seventh case we need $n \geq 5$. In each case, though, the given formula works for smaller positive values of n since it is equal to 0 in each case. Note also that we say $b = 1$ in the first case since there is exactly 1 way of placing 0s in all of the remaining $n-1$ positions.)

$f(n)$ is then the sum of the expressions in the last column of this table, and so

$$f(n) = \frac{n^4+6n^3+11n^2+6n}{24} = \frac{n(n+1)(n+2)(n+3)}{24}$$

as required.

Method 2

First, we create a correspondence between each integer with n digits and whose digits have

a sum of 5 and an arrangement of five 1s and $(n-1)$ Xs that begins with a 1.
We can then count these integers by counting the arrangements.
Starting with such an integer, we write down an arrangement of the above type using the following rule:

> The number of 1s to the left of the first X is the first digit of the number, the number of 1s between the first X and second X is the second digit of the number, and so on, with the number of 1s to the right of the $(n-1)$st X representing the nth digit of the number.

For example, the integer 1010020001 would correspond to 1XX1XXX11XXXX1.
In this way, each such integer gives an arrangement of the above type.
Similarly, each arrangement of this type can be associated back to a unique integer with the required properties by counting the number of 1s before the first X and writing this down as the leftmost digit, counting the number of 1s between the first and second Xs and writing this down as the second digit, and so on. Since a total of five 1s are used, then each arrangement corresponds with an integer with n digits whose digits have a sum of 5.
Therefore, there is a one-to-one correspondence between the integers and arrangements with the desired properties.
Thus, $f(n)$, which equals the number of such integers, also equals the number of such arrangements.
To count the number of such arrangements, we note that there are four 1s and $n-1$ Xs to arrange in the final $4+(n-1)=n+3$ positions, since the first position is occupied by a 1.
There are $\binom{n+3}{4}$ ways to choose the positions of the remaining four 1s, and so $\binom{n+3}{4}$ arrangements.
Thus, $f(n)=\binom{n+3}{4}=\dfrac{(n+3)!}{4!(n-1)!}=\dfrac{(n+3)(n+2)(n+1)(n)}{4!}=\dfrac{n(n+1)(n+2)(n+3)}{24}$.

Next, we need to determine the positive integers n between 1 and 2014, inclusive, for which the units digit of $f(n)$ is 1.
Now $f(n)=\dfrac{n(n+1)(n+2)(n+3)}{24}$ is an integer for all positive integers n, since it is counting the number of things with a certain property.
If the units digit of n is 0 or 5, then n is a multiple of 5.
If the units digit of n is 2 or 7, then $n+3$ is a multiple of 5.
If the units digit of n is 3 or 8, then $n+2$ is a multiple of 5.
If the units digit of n is 4 or 9, then $n+1$ is a multiple of 5.
Thus, if the units digit of n is 0, 2, 3, 4, 5, 7, 8, or 9, then $n(n+1)(n+2)(n+3)$ is a multiple of 5 and so $f(n)=\dfrac{n(n+1)(n+2)(n+3)}{24}$ is a multiple of 5, since the denominator contains no factors of 5 that can divide the factor from the numerator.
Therefore, if the units digit of n is 0, 2, 3, 4, 5, 7, 8, or 9, then $f(n)$ is divisible by 5, and so cannot have a units digit of 1.

So we consider the cases where n has a units digit of 1 or of 6; these are the only possible values of n for which $f(n)$ can have a units digit of 1.
We note that $3f(n)=\dfrac{n(n+1)(n+2)(n+3)}{8}$, which is a positive integer for all positive integers n.

Also, we note that if $f(n)$ has units digit 1, then $3f(n)$ has units digit 3, and if $3f(n)$ has units digit 3, then $f(n)$ must have units digit 1.
Therefore, determining the values of n for which $f(n)$ has units digit 1 is equivalent to determining the values of n for which $\dfrac{n(n+1)(n+2)(n+3)}{8}$ has units digit 3.

We consider the integers n in groups of 40. (Intuitively, we do this because the problem seems to involve multiples of 5 and multiples of 8, and $5 \times 8 = 40$.)
If n has units digit 1, then $n = 40k + 1$ or $n = 40k + 11$ or $n = 40k + 21$ or $n = 40k + 31$ for some integer $k \geq 0$.
If n has units digit 6, then $n = 40k + 6$ or $n = 40k + 16$ or $n = 40k + 26$ or $n = 40k + 36$ for some integer $k \geq 0$.

If $n = 40k + 1$, then

$$\begin{aligned} 3f(n) &= \frac{n(n+1)(n+2)(n+3)}{8} \\ &= \frac{(40k+1)(40k+2)(40k+3)(40k+4)}{8} \\ &= (40k+1)(20k+1)(40k+3)(10k+1) \end{aligned}$$

The units digit of $40k + 1$ is 1, the units digit of $20k + 1$ is 1, the units digit of $40k + 3$ is 3, and the units digit of $10k + 1$ is 1, so the units digit of the product is the units digit of $(1)(1)(3)(1)$ or 3.
In a similar way, we treat the remaining seven cases and summarize all eight cases in a chart:

n	$3f(n)$ simplified	Units digit of $3f(n)$
$40k+1$	$(40k+1)(20k+1)(40k+3)(10k+1)$	3
$40k+11$	$(40k+11)(10k+3)(40k+13)(20k+7)$	3
$40k+21$	$(40k+21)(20k+11)(40k+23)(10k+6)$	8
$40k+31$	$(40k+31)(10k+8)(40k+33)(20k+17)$	8
$40k+6$	$(20k+3)(40k+7)(10k+2)(40k+9)$	8
$40k+16$	$(10k+4)(40k+17)(20k+9)(40k+19)$	8
$40k+26$	$(20k+13)(40k+27)(10k+7)(40k+29)$	3
$40k+36$	$(10k+9)(40k+37)(20k+19)(40k+39)$	3

(Note that, for example, when $n = 40k + 16$, the simplified version of $3f(n)$ is $(10k + 4)(40k + 17)(20k + 9)(40k + 19)$, so the units digit of $3f(n)$ is the units digit of $(4)(7)(9)(9)$ which is the units digit of 2268, or 8.)

Therefore, $f(n)$ has units digit 1 whenever $n = 40k + 1$ or $n = 40k + 11$ or $n = 40k + 26$ or $n = 40k + 36$ for some integer $k \geq 0$.
There are 4 such integers n between each pair of consecutive multiples of 40.
Since $2000 = 50 \times 40$, then 2000 is the 50th multiple of 40, so there are $50 \times 4 = 200$ integers n less than 2000 for which the units digit of $f(n)$ is 1.
Between 2000 and 2014, inclusive, there are two additional integers: $n = 40(50)+1 = 2001$ and $n = 40(50) + 11 = 2011$.
In total, 202 of the integers $f(1), f(2), \ldots, f(2014)$ have a units digit of 1.

10. Throughout this solution, we use "JB" to represent "jelly bean" or "jelly beans".
We use "T1" to represent "Type 1 move", "T2" to represent "Type 2 move", and so on.
We use "P0" to represent "position 0", "P1" to represent "position 1", and so on.
We represent the positions of the JB initially or after a move using an ordered tuple of non-negative integers representing the number of JB at P0, P1, P2, etc. For example, the tuple $(0, 0, 1, 2, 1)$ would represent 0 JB at P0, 0 JB at P1, 1 JB at P2, 2 JB at P3, and 1 JB at P4.

(a) To begin, we work backwards from the final state $(0, 0, 0, 0, 0, 1)$.
The only move that could have put 1 JB at P5 is 1 T5.
Undoing this move removes 1 JB from P5 and adds 1 JB at P4 and 1 JB at P3, giving $(0, 0, 0, 1, 1, 0)$.
The only move that could have put 1 JB at P4 is 1 T4.
Undoing this move removes 1 JB from P4 and adds 1 JB at P2 and 1 JB at P3, giving $(0, 0, 1, 2, 0, 0)$.
The only moves that could put 2 JB at P3 are 2 T3s.
Undoing these moves removes 2 JB from P3, adds 2 JB at P1 and 2 JB at P2, giving $(0, 2, 3, 0, 0, 0)$.
The only moves that could put 3 JB at P2 are 3 T2s.
Undoing these moves gives $(3, 5, 0, 0, 0, 0)$.
The only moves that could put 5 JB at P1 are 5 T1s.
Undoing these moves removes 5 JB from P1 and adds 10 JB at P0, giving $(13, 0, 0, 0, 0, 0)$.
Therefore, starting with $N = 13$ JB at P0 allows Fiona to win the game by making all of the moves as above in the reverse order.
In particular, from $(13, 0, 0, 0, 0, 0)$, 5 T1s gives $(3, 5, 0, 0, 0, 0)$, then 3 T2s give $(0, 2, 3, 0, 0, 0)$, then 2 T3s give $(0, 0, 1, 2, 0, 0)$, then 1 T4 gives $(0, 0, 0, 1, 1, 0)$, then 1 T5 gives $(0, 0, 0, 0, 0, 1)$, as required.

(b) Initial Set-up
First, we note that when Fiona starts with N JB (for some fixed positive integer N), then the game finishes in at most $N - 1$ moves (since she eats exactly one JB on each move).
Second, we note that the positions of the JB in the final state as well as at any intermediate state (that is, after some number of moves) must be in the list P0, P1, ..., P$(N - 1)$, since each JB can move at most 1 position to the right on any given move, so no JB can move more than $N - 1$ positions to the right in at most $N - 1$ moves.
This means that, starting with N JB, any state can be described using an N-tuple $(a_0, a_1, \ldots, a_{N-2}, a_{N-1})$, where a_i represents the number of JB at Pi in that state.

Introduction of Fibonacci Sequence and Important Fact #1 (IF1)
We define the Fibonacci sequence by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.
The initial number of JB (N) and the number of JB at various positions are connected using the Fibonacci sequence in the following way.
At any state between the starting state (N JB at P0) and the final state, if there are a_i JB at Pi for each i from 0 to $N - 1$, then

$$N = a_0F_2 + a_1F_3 + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1} \qquad (*)$$

This is true because:

- It is true for the starting state, since here $(a_0, a_1, \ldots, a_{N-2}, a_{N-1}) = (N, 0, \ldots, 0, 0)$ and $F_2 = 1$, so the right side of $(*)$ equals $N(1) + 0$ or N
- A T1 does not change the value of the right side of $(*)$: Since a T1 changes the state $(a_0, a_1, a_2, \ldots, a_{N-2}, a_{N-1})$ to $(a_0 - 2, a_1 + 1, a_2, \ldots, a_{N-2}, a_{N-1})$, the right side of $(*)$

changes from

$$a_0F_2 + a_1F_3 + a_2F_4 + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1}$$

to

$$(a_0 - 2)F_2 + (a_1 + 1)F_3 + a_2F_4 + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1}$$

which is a difference of $-2F_2 + F_3 = -2(1) + 2 = 0$.

- A Ti for $i \geq 2$ does not change the value of the right side of $(*)$: Since a Ti changes the state $(a_0, a_1, \ldots, a_{i-2}, a_{i-1}, a_i, \ldots, a_{N-2}, a_{N-1})$ to

$$(a_0, a_1, \ldots, a_{i-2} - 1, a_{i-1} - 1, a_i + 1, \ldots, a_{N-2}, a_{N-1})$$

the right side of $(*)$ changes from

$$a_0F_2 + a_1F_3 + \cdots + a_{i-2}F_i + a_{i-1}F_{i+1} + a_iF_{i+2} + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1}$$

to

$$a_0F_2 + a_1F_3 + \cdots + (a_{i-2} - 1)F_i + (a_{i-1} - 1)F_{i+1} + (a_i + 1)F_{i+2} + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1}$$

which is a difference of $-F_i - F_{i+1} + F_{i+2} = 0$ since $F_{i+2} = F_{i+1} + F_i$.

This tells us that the value of the right side of $(*)$ starts at N and does not change on any subsequent move.
Therefore, at any state $(a_0, a_1, \ldots, a_{N-2}, a_{N-1})$ after starting with N JB at P0, it is true that

$$N = a_0F_2 + a_1F_3 + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1} \qquad (*)$$

To show that there is only one possible final state when Fiona wins the game, we assume that there are two possible winning final states starting from N JB and show that these in fact must be the same state.

Important Fact #2 (IF2)
To do this, we prove a property of Fibonacci numbers that will allow us to show that two sums of three or fewer non-consecutive Fibonacci numbers cannot be equal if the Fibonacci numbers used in each sum are not the same:

If x, y, z are positive integers with $2 \leq x < y < z$ and no pair of x, y, z are consecutive integers, then $F_z < F_y + F_z < F_x + F_y + F_z < F_{z+1}$.

Since each Fibonacci number is a positive integer, then $F_z < F_y + F_z < F_x + F_y + F_z$, so we must prove that $F_x + F_y + F_z < F_{z+1}$:

Since no two of x, y, z are consecutive and $x < y < z$, then $y < z - 1$.
Since y and z are positive integers, then $y \leq z - 2$.
Also, $x < y - 1 \leq z - 3$.
Since x and z are integers with $x < z - 3$, then $x \leq z - 4$.
Since the Fibonacci sequence is increasing from F_2 onwards, then

$$F_x + F_y + F_z \leq F_{z-4} + F_{z-2} + F_z < F_{z-3} + F_{z-2} + F_z = F_{z-1} + F_z = F_{z+1}$$

Since there is a "$<$" in this chain of inequalities and equalities, then we obtain that $F_x + F_y + F_z < F_{z+1}$, as required.

Completing the Proof

Recall from the statement of the problem that a winning state consists of three or fewer JB, each at a distinct position and no two at consecutive integer positions.

Suppose that, starting from N JB at P0, in a first winning final state with $a_d = 1$, each of a_b and a_c equal to 0 or 1 and all other $a_i = 0$, and in a second winning final state with $a_D = 1$, each of a_B and a_C equal to 0 or 1 and all other $a_i = 0$.

From IF1, this gives $N = a_bF_{b+2} + a_cF_{c+2} + F_{d+2}$ and $N = a_BF_{B+2} + a_CF_{C+2} + F_{D+2}$, and so $a_bF_{b+2} + a_cF_{c+2} + F_{d+2} = a_BF_{B+2} + a_CF_{C+2} + F_{D+2}$.

Starting from this last equation, we remove any common Fibonacci numbers from both sides. (Recall that each term on each side is either 0 or a Fibonacci number, and Fibonacci numbers on the same side are distinct.)

If there are no Fibonacci numbers remaining on each side, then the winning final states are the same, as required.

What happens if there are Fibonacci numbers remaining on either side? In this case, there must be Fibonacci numbers on each side, as otherwise we would have 0 equal to a non-zero number.

Suppose that the largest Fibonacci number remaining on the LS is F_k and the largest Fibonacci number remaining on the RS is F_m.

Since we have removed the common elements, then $k \neq m$, so we may assume that $k < m$; since k and m are integers, then $k \leq m - 1$.

Note that the RS must be greater than or equal to F_m, since it includes at least F_m.

Since the LS consists of at most three Fibonacci numbers, which are non-consecutive (since b, c, d are non-consecutive) and the largest of which is F_k, then IF2 tells us that the LS is less than F_{k+1}.

Since $k + 1 \leq m$, then the LS is less than F_m.

Since the LS is less than F_m and the RS is greater than or equal to F_m, we have a contradiction, since they are supposed to be equal.

Therefore, our assumption that Fibonacci numbers are left after removing the common numbers from each side is false.

In other words, the positions of the JB in each of the winning final states are the same, so there is indeed only one possible winning final state.

Therefore, if Fiona can win the game, then there is only one possible final state.

(c) From the statement of the problem and IF1, we know that Fiona can win the game starting with N JB at P0 only if N is equal to the sum of at most three distinct non-consecutive Fibonacci numbers.

To determine the closest positive integer N to 2014 for which Fiona can win the game, we can determine the closet positive integer to 2014 that can be written as the sum of at most three distinct Fibonacci numbers, no two of which are consecutive.

We write out terms in the Fibonacci sequence until we reach a term larger than 2014:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584$$

We note that $1597 + 377 + 34 = 2008$, which is 6 away from 2014. We will show that there we cannot achieve an answer closer to 2014. That is, we will show that we cannot achieve any of the integers from 2009 to 2019, inclusive.

Suppose that an integer from 2009 to 2019, inclusive, can be achieved.

The Fibonacci number 2584 cannot be included in our sum, as the sum would be too large.

If our sum includes no Fibonacci number larger than 987, then our sum is at most $987 + 377 + 144 = 1508$, which is not large enough.

Therefore, 1597 must be included in a sum equal to an integer in the range 2009 to 2019, inclusive.

The remaining 0, 1 or 2 Fibonacci numbers must have a sum in the range $2009 - 1597 = 412$ to $2019 - 1597 = 422$, inclusive.

No Fibonacci number larger than 377 can be used, otherwise the remaining sum would be too large.

If the remaining sum uses no Fibonacci number larger than 233, the sum is at most $233 + 89 = 322$, which is not in the desired range.

Therefore, 377 must be included in the remaining sum.

The remaining 0 or 1 Fibonacci numbers must have a sum in the range $412 - 377 = 35$ to $422 - 377 = 45$, inclusive.

There is no Fibonacci number in this range, so we cannot make a sum of at most three distinct, non-consecutive numbers that is closer to 2014 than 2008.

Note that $2008 = 1597 + 377 + 34$. Since $F_9 = 34$, $F_{14} = 377$ and $F_{17} = 1597$, the corresponding winning position would be 1 JB at each of P7, P12 and P15.

To complete our proof, we must show that we can actually achieve this final state:

> We start with the final state consisting of 1 JB at each of P7, P12 and P15 and play the game backwards as we did in (a).
>
> Since there is 1 JB at P15, it must have come from a T15.
>
> Undoing this move, we obtain a state consisting of 1 JB at each of P7, P12, P13 and P14. Note that the rightmost JB is now at P14.
>
> Since there is 1 JB at P14, it must have come from a T14.
>
> We undo this move and continue to undo moves that remove a JB from the rightmost position remaining at each step. This process will eventually move all of the JB back to P0.
>
> To win the game starting with $N = F_9 + F_{14} + F_{17}$, Fiona then uses all of these moves in the opposite order, in a similar way to the method in (a).
>
> Thus, Fiona can achieve the winning final state of 1 JB at each of P7, P12 and P15.

Therefore, if $N = F_9 + F_{14} + F_{17}$, then Fiona can win the game.

Thus, $N = 2008$ is the closest integer to 2014 for which Fiona can start with N JB at P0 and win the game.