The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Tuesday, April 15, 2014
(in North America and South America)

Wednesday, April 16, 2014
(outside of North America and South America)

UNIVERSITY OF WATERLOO   WATERLOO MATHEMATICS   Deloitte.

Do not open this booklet until instructed to do so.

Time: $2\frac{1}{2}$ hours
Calculators are permitted, provided they are non-programmable and without graphic displays.

Number of questions: 10
Each question is worth 10 marks

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, http://www.cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

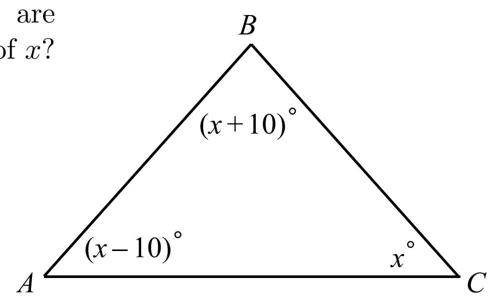
TIPS:	
	1. Please read the instructions on the front cover of this booklet.
	2. Write all answers in the answer booklet provided.
	3. For questions marked (light bulb), place your answer in the appropriate box in the answer booklet and **show your work**.
	4. For questions marked (pencil), provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
	5. Diagrams are *not* drawn to scale. They are intended as aids only.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth, grade, and sex, on the Student Information Form, and that you have answered the question about eligibility.

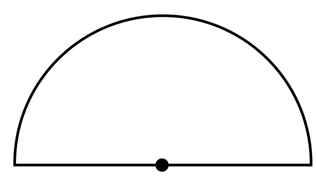
1. (a) What is the value of $\dfrac{\sqrt{16}+\sqrt{9}}{\sqrt{16+9}}$?

   (b) In the diagram, the angles of $\triangle ABC$ are shown in terms of x. What is the value of x?



   (c) Lisa earns two times as much per hour as Bart. Lisa works 6 hours and Bart works 4 hours. They earn \$200 in total. How much does Lisa earn per hour?

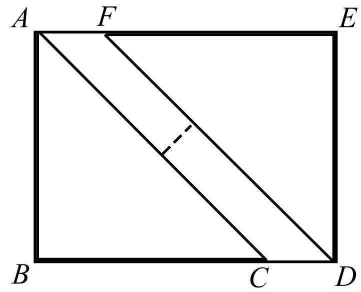
2. (a) The semi-circular region shown has radius 10. What is the perimeter of the region?



   (b) The parabola with equation $y = 10(x+2)(x-5)$ intersects the x-axis at points P and Q. What is the length of line segment PQ?

   (c) The line with equation $y = 2x$ intersects the line segment joining $C(0, 60)$ and $D(30, 0)$ at the point E. Determine the coordinates of E.

3. (a) Jimmy is baking two large identical triangular cookies, $\triangle ABC$ and $\triangle DEF$. Each cookie is in the shape of an isosceles right-angled triangle. The length of the shorter sides of each of these triangles is 20 cm. He puts the cookies on a rectangular baking tray so that A, B, D, and E are at the vertices of the rectangle, as shown. If the distance between parallel sides AC and DF is 4 cm, what is the width BD of the tray?



(b) Determine all values of x for which $\dfrac{x^2+x+4}{2x+1}=\dfrac{4}{x}$.

4. (a) Determine the number of positive divisors of 900, including 1 and 900, that are perfect squares. (A *positive divisor* of 900 is a positive integer that divides exactly into 900.)

(b) Points $A(k,3)$, $B(3,1)$ and $C(6,k)$ form an isosceles triangle. If $\angle ABC=\angle ACB$, determine all possible values of k.

5. (a) A chemist has three bottles, each containing a mixture of acid and water:

- bottle A contains 40 g of which 10% is acid,
- bottle B contains 50 g of which 20% is acid, and
- bottle C contains 50 g of which 30% is acid.

She uses some of the mixture from each of the bottles to create a mixture with mass 60 g of which 25% is acid. Then she mixes the remaining contents of the bottles to create a new mixture. What percentage of the new mixture is acid?

(b) Suppose that x and y are real numbers with $3x+4y=10$. Determine the minimum possible value of x^2+16y^2.

6. (a) A bag contains 40 balls, each of which is black or gold. Feridun reaches into the bag and randomly removes two balls. Each ball in the bag is equally likely to be removed. If the probability that two gold balls are removed is $\frac{5}{12}$, how many of the 40 balls are gold?

(b) The geometric sequence with n terms $t_1,t_2,\ldots,t_{n-1},t_n$ has $t_1t_n=3$. Also, the product of all n terms equals 59 049 (that is, $t_1t_2\cdots t_{n-1}t_n=59\,049$). Determine the value of n.

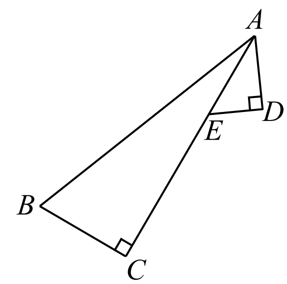
(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

7. (a) If $\dfrac{(x-2013)(y-2014)}{(x-2013)^2+(y-2014)^2} = -\dfrac{1}{2}$, what is the value of $x+y$?

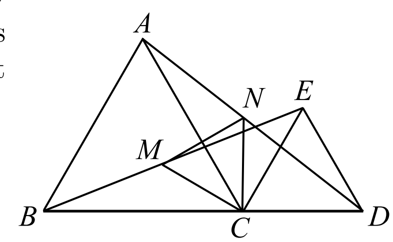
(b) Determine all real numbers x for which

$$(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10\,000$$

8. (a) In the diagram, $\angle ACB = \angle ADE = 90^\circ$. If $AB = 75$, $BC = 21$, $AD = 20$, and $CE = 47$, determine the exact length of BD.



(b) In the diagram, C lies on BD. Also, $\triangle ABC$ and $\triangle ECD$ are equilateral triangles. If M is the midpoint of BE and N is the midpoint of AD, prove that $\triangle MNC$ is equilateral.



9. (a) Without using a calculator, determine positive integers m and n for which

$$\sin^6 1^\circ + \sin^6 2^\circ + \sin^6 3^\circ + \cdots + \sin^6 87^\circ + \sin^6 88^\circ + \sin^6 89^\circ = \frac{m}{n}$$

(The sum on the left side of the equation consists of 89 terms of the form $\sin^6 x^\circ$, where x takes each positive integer value from 1 to 89.)

(b) Let $f(n)$ be the number of positive integers that have exactly n digits and whose digits have a sum of 5. Determine, with proof, how many of the 2014 integers $f(1), f(2), \ldots, f(2014)$ have a units digit of 1.

10. Fiona plays a game with jelly beans on the number line. Initially, she has N jelly beans, all at position 0. On each turn, she must choose one of the following moves:

- Type 1: She removes two jelly beans from position 0, eats one, and puts the other at position 1.
- Type i, where i is an integer with $i \geq 2$: She removes one jelly bean from position $i - 2$ and one jelly bean from position $i - 1$, eats one, and puts the other at position i.

The positions of the jelly beans when no more moves are possible is called the *final state*. Once a final state is reached, Fiona is said to have won the game if there are at most three jelly beans remaining, each at a distinct position and no two at consecutive integer positions. For example, if $N = 7$, Fiona wins the game with the sequence of moves

Type 1, Type 1, Type 2, Type 1, Type 3

which leaves jelly beans at positions 1 and 3. A different sequence of moves starting with $N = 7$ might not win the game.

(a) Determine an integer N for which it is possible to win the game with one jelly bean left at position 5 and no jelly beans left at any other position.

(b) Suppose that Fiona starts the game with a fixed unknown positive integer N. Prove that if Fiona can win the game, then there is only one possible final state.

(c) Determine, with justification, the closest positive integer N to 2014 for which Fiona can win the game.

2014 Euclid Contest (English)

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For students...

Thank you for writing the 2014 Euclid Contest!
In 2013, more than 17 000 students from around the world registered to write the Euclid Contest.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2014 Canadian Senior Mathematics Contest, which will be written in November 2014.

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