The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Pascal Contest

(Grade 9)

Thursday, February 21, 2013
(in North America and South America)

Friday, February 22, 2013
(outside of North America and South America)

Solutions

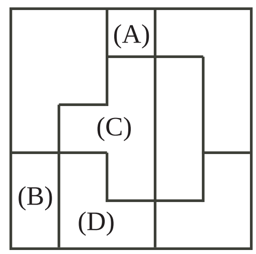
1. Simplifying first inside the brackets, $(4 + 44 + 444) \div 4 = 492 \div 4 = 123$.

   Answer: (B)

2. Since Jing purchased 8 identical items and the total cost was \$26, then to obtain the cost per item, she divides the total cost by the number of items.
   Thus, the answer is $26 \div 8$, or (A).

   Answer: (A)

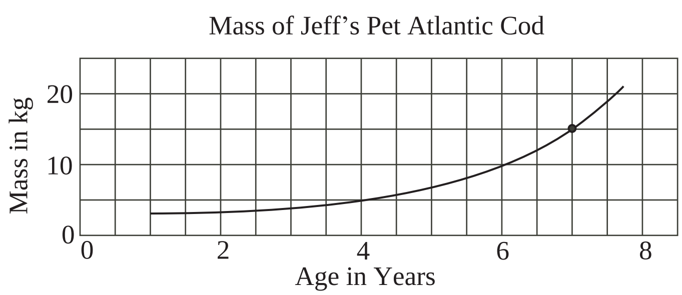
3. Each of pieces (A), (B), (C), and (D) occurs in the diagram.



   Piece (E) does not occur in the diagram.

   Answer: (E)

4. A mass of 15 kg is halfway between 10 kg and 20 kg on the vertical axis.
   The point where the graph reaches 15 kg is halfway between 6 and 8 on the horizontal axis.



   Therefore, the cod is 7 years old when its mass is 15 kg.

   Answer: (B)

5. Expanding and simplifying,

$$1^3 + 2^3 + 3^3 + 4^3 = 1 \times 1 \times 1 + 2 \times 2 \times 2 + 3 \times 3 \times 3 + 4 \times 4 \times 4 = 1 + 8 + 27 + 64 = 100$$

   Since $100 = 10^2$, then $1^3 + 2^3 + 3^3 + 4^3 = 10^2$.

   Answer: (C)

6. Since Erin walks $\frac{3}{5}$ of the way home in 30 minutes, then she walks $\frac{1}{5}$ of the way at the same rate in 10 minutes.
   She has $1 - \frac{3}{5} = \frac{2}{5}$ of the way left to walk. This is twice as far as $\frac{1}{5}$ of the way.
   Since she continues to walk at the same rate and $\frac{1}{5}$ of the way takes her 10 minutes, then it takes her $2 \times 10 = 20$ minutes to walk the rest of the way home.

   Answer: (B)

7. Simplifying, $(\sqrt{100} + \sqrt{9}) \times (\sqrt{100} - \sqrt{9}) = (10 + 3) \times (10 - 3) = 13 \times 7 = 91$.

   Answer: (A)

8. Since $PQRS$ is a rectangle, then $QR = PS = 6$.
   Therefore, $UR = QR - QU = 6 - 2 = 4$.
   Since $PQRS$ is a rectangle and TU is perpendicular to QR, then TU is parallel to and equal to SR, so $TU = 3$.
   By the Pythagorean Theorem, since $TR > 0$, then

   $$TR = \sqrt{TU^2 + UR^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

   Thus, $TR = 5$. (We could also notice the 3-4-5 right-angled triangle.)

   Answer: (C)

9. Since Owen uses an average of 1 L to drive 12.5 km, then it costs Owen \$1.20 in gas to drive 12.5 km.
   To drive 50 km, he drives 12.5 km a total of $50 \div 12.5 = 4$ times.
   Therefore, it costs him $4 \times \$1.20 = \$4.80$ in gas to drive 50 km.

   Answer: (A)

10. There are six times that can be made using each of the digits 2, 3 and 5 exactly once: 2:35, 2:53, 3:25, 3:52, 5:23, and 5:32.
    The first of these that occurs after 3:52 is 5:23.
    From 3:52 to 4:00, 8 minutes pass.
    From 4:00 to 5:00, 60 minutes pass.
    From 5:00 to 5:23, 23 minutes pass.
    Therefore, from 3:52 to 5:23, which is the next time that uses the digits 2, 3, and 5 each exactly once, a total of $8 + 60 + 23 = 91$ minutes pass.

    Answer: (D)

11. Since the sequence repeats every 4 symbols and since $13 \times 4 = 52$, then the 52nd symbol is the last symbol in a sequence of 4 symbols.
    Also, the first 52 symbols represent 13 sequences of these 4 symbols.
    Each sequence of 4 symbols includes 2 $\heartsuit$s, so the first 52 symbols include $13 \times 2 = 26$ $\heartsuit$s.
    Finally, the 53rd symbol in the pattern is the first of a sequence of 4, so is also $\heartsuit$.
    Therefore, the first 53 symbols include $26 + 1 = 27$ $\heartsuit$s.

    Answer: (C)

12. Since $x = 11$, $y = -8$ and $2x - 3z = 5y$, then $2 \times 11 - 3z = 5 \times (-8)$ or $22 - 3z = -40$.
    Therefore, $3z = 22 + 40 = 62$ and so $z = \frac{62}{3}$.

    Answer: (D)

13. The original set contains 11 elements whose sum is 66.
    When one number is removed, there will be 10 elements in the set.
    For the average of these elements to be 6.1, their sum must be $10 \times 6.1 = 61$.
    Since the sum of the original 11 elements is 66 and the sum of the remaining 10 elements is 61, then the element that must be removed is $66 - 61 = 5$.

    Answer: (B)

14. Since $\angle QTS = 76^\circ$ and $\triangle QTS$ has $QS = QT$, then $\angle QST = \angle QTS = 76^\circ$.
Since the angles in $\triangle QTS$ add to 180°, then

$$\angle SQT = 180^\circ - \angle QTS - \angle QST = 180^\circ - 76^\circ - 76^\circ = 28^\circ$$

Since PQR is a straight line segment, then $\angle PQS + \angle SQT + \angle TQR = 180^\circ$.
Thus, $x^\circ + 28^\circ + 3x^\circ = 180^\circ$.
This gives $4x + 28 = 180$ or $4x = 152$ and so $x = 38$.

Answer: (B)

15. We note that $64 = 4 \times 4 \times 4$.
Thus, $64^2 = 64 \times 64 = 4 \times 4 \times 4 \times 4 \times 4 \times 4$.
Since $4^n = 64^2$, then $4^n = 4 \times 4 \times 4 \times 4 \times 4 \times 4$ and so $n = 6$.

Answer: (C)

16. *Solution 1*
If $x = 1$, then $3x + 1 = 4$, which is an even integer.
In this case, the five given choices are

(A) $x + 3 = 4$ (B) $x - 3 = -2$ (C) $2x = 2$ (D) $7x + 4 = 11$ (E) $5x + 3 = 8$

Of these, the only odd integer is (D). Therefore, since $x = 1$ satisfies the initial criteria, then (D) must be the correct answer as the result must be true no matter what integer value of x is chosen that makes $3x + 1$ even.

Solution 2
If x is an integer for which $3x + 1$ is even, then $3x$ is odd, since it is 1 less than an even integer.
If $3x$ is odd, then x must be odd (since if x is even, then $3x$ would be even).
If x is odd, then $x + 3$ is even (odd plus odd equals even), so (A) cannot be correct.
If x is odd, then $x - 3$ is even (odd minus odd equals even), so (B) cannot be correct.
If x is odd, then $2x$ is even (even times odd equals even), so (C) cannot be correct.
If x is odd, then $7x$ is odd (odd times odd equals odd) and so $7x + 4$ is odd (odd plus even equals odd).
If x is odd, then $5x$ is odd (odd times odd equals odd) and so $5x + 3$ is even (odd plus odd equals even), so (E) cannot be correct.
Therefore, the one expression which must be odd is $7x + 4$.

Answer: (D)

17. Since 40% of the songs on the updated playlist are Country, then the remaining $100\% - 40\%$ or 60% must be Hip Hop or Pop songs.
Since the ratio of Hip Hop songs to Pop songs does not change, then 65% of this remaining 60% must be Hip Hop songs.
Overall, this is $65\% \times 60\% = 0.65 \times 0.6 = 0.39 = 39\%$ of the total number of songs on the playlist.

Answer: (E)

18. The area of the shaded region is equal to the area of square $PQRS$ minus the combined areas of the four unshaded regions inside the square.
Since square $PQRS$ has side length 2, its area is $2^2 = 4$.
Since $PQRS$ is a square, then the angle at each of P, Q, R, and S is 90°.

Since each of P, Q, R, and S is the centre of a circle with radius 1, then each of the four unshaded regions inside the square is a quarter circle of radius 1. (A central angle of 90° gives a quarter of a circle.)
Thus, the combined areas of the four unshaded regions inside the square equals four quarters of a circle of radius 1, or the area of a whole circle of radius 1.
This area equals $\pi(1)^2 = \pi$.
Therefore, the shaded region equals $4 - \pi$.

ANSWER: (E)

19. Since the flag shown is rectangular, then its total area is its height multiplied by its width, or $h \times 2h = 2h^2$.
Since the flag is divided into seven stripes of equal height and each stripe has equal width, then the area of each stripe is the same.
Since the four shaded strips have total area 1400 cm^2, then the area of each strip is $1400 \div 4 = 350\text{ cm}^2$.
Since the flag consists of 7 strips, then the total area of the flag is $350\text{ cm}^2 \times 7 = 2450\text{ cm}^2$.
Since the flag is h by $2h$, then $2h^2 = 2450\text{ cm}^2$ or $h^2 = 1225\text{ cm}^2$.
Therefore, $h = \sqrt{1225\text{ cm}^2} = 35$ cm (since $h > 0$).
The height of the flag is 35 cm.

ANSWER: (C)

20. We make a chart that shows the possible combinations of the number that Sam rolls and the number that Tyler rolls. Since Sam rolls a fair four-sided die and Tyler rolls a fair six-sided die, then there are 4 possible numbers that Sam can roll and 6 possible numbers that Tyler can roll and so there are $4 \times 6 = 24$ equally likely combinations in total. In the chart, we put a Y when Sam's roll is larger than Tyler's and an N otherwise.

		Tyler's roll					
		1	2	3	4	5	6
Sam's roll	1	N	N	N	N	N	N
	2	Y	N	N	N	N	N
	3	Y	Y	N	N	N	N
	4	Y	Y	Y	N	N	N

Since there are 24 equally likely possibilities and Sam's roll is larger in 6 of these, then the probability that Sam's roll is larger than Tyler's is $\frac{6}{24} = \frac{1}{4}$.

ANSWER: (E)

21. We begin by factoring the given integer into prime factors.
Since 636 405 ends in a 5, it is divisible by 5, so

$$636\,405 = 5 \times 127\,281$$

Since the sum of the digits of 127 281 is a multiple of 3, then it is a multiple of 3, so

$$636\,405 = 5 \times 3 \times 42\,427$$

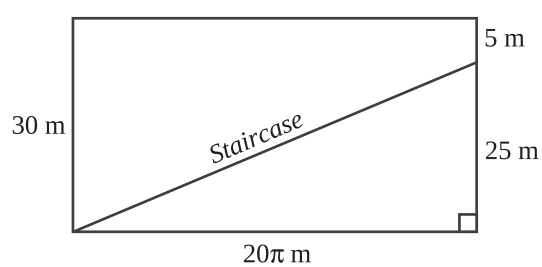
The new quotient (42 427) is divisible by 7 (can you see this without using a calculator?), which gives

$$636\,405 = 5 \times 3 \times 7 \times 6061$$

We can proceed by systematic trial and error to see if 6061 is divisible by 11, 13, 17, 19, and so on. After some work, we can see that $6061 = 11 \times 551 = 11 \times 19 \times 29$.
Therefore, $636\,405 = 3 \times 5 \times 7 \times 11 \times 19 \times 29$.
We want to rewrite this as the product of three 2-digit numbers.
Since $3 \times 5 \times 7 = 105$ which has three digits, and the product of any three of the six prime factors of $636\,405$ is at least as large as this, then we cannot take the product of three of these prime factors to form a two-digit number.
Thus, we have to combine the six prime factors in pairs.
The prime factor 29 cannot be multiplied by any prime factor larger than 3, since $29 \times 3 = 87$ which has two digits, but $29 \times 5 = 145$, which has too many digits.
This gives us $636\,405 = 87 \times 5 \times 7 \times 11 \times 19$.
The prime factor 19 can be multiplied by 5 (since $19 \times 5 = 95$ which has two digits) but cannot be multiplied by any prime factor larger than 5, since $19 \times 7 = 133$, which has too many digits.
This gives us $636\,405 = 87 \times 95 \times 7 \times 11 = 87 \times 95 \times 77$.
The sum of these three 2-digit divisors is $87 + 95 + 77 = 259$.

Answer: (A)

22. To calculate the total distance, we add the length of the vertical ladder (5 m) to the length of the spiral staircase.
The spiral staircase wraps once around the tower.
Since the tower has radius 10 m, then its circumference is $2 \times \pi \times 10 = 20\pi$ m.
We can thus "unwrap" the outside of the tower to form a rectangle of width 20π m and height 30 m.
Since the staircase has a constant slope, then the staircase becomes a straight line on the unwrapped tower.
Since the ladder accounts for the final 5 m of the height of the tower and the tower has total height 30 m, then the top of the spiral staircase is $30 - 5 = 25$ m above its base.



We can thus calculate the length of the staircase (which is positive), using the Pythagorean Theorem, to be

$$\sqrt{(20\pi \text{ m})^2 + (25 \text{ m})^2} \approx 67.62 \text{ m}$$

The total distance along the staircase and up the ladder is approximately $5 + 67.62 \approx 72.62$ m. Of the given choices, this is closest to 72.6 m.

Answer: (A)

23. Suppose that the five distinct numbers that Joshua chooses are V, W, X, Y, Z, and that $V < W < X < Y < Z$.
We want to assign these to p, q, r, s, t so that $p < s$ and $q < s$ and $r < t$ and $s < t$.
First, we note that t must be the largest of p, q, r, s, t. This is because $r < t$ and $s < t$, and because $p < s$ and $q < s$, we get $p < s < t$ and $q < s < t$, so $p < t$ and $q < t$.

Since t is the largest, then Z must be t.

Now neither p nor q can be the second largest of the numbers (which is Y), since p and q are both smaller than s and t.
Therefore, there are two cases: $Y = r$ or $Y = s$.

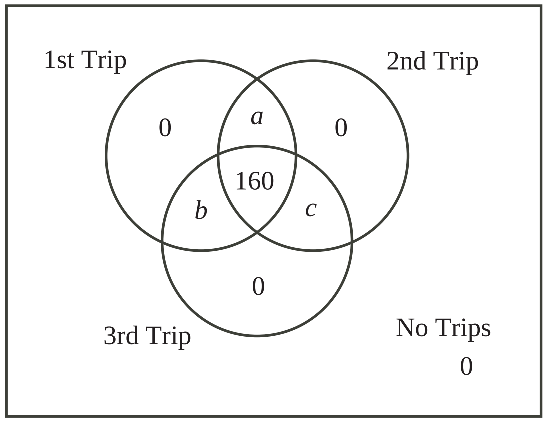
Case 1: $Y = r$
We have $Y = r$ and $Z = t$.
This leaves V, W, X (which satisfy $V < W < X$) to be assigned to p, q, s (which satisfy $p < s$ and $q < s$).
Since X is the largest of V, W, X and s is the largest of p, q, s, then $X = s$.
This leaves V, W to be assigned to p, q.
Since there is no known relationship between p and q, then there are 2 possibilities: either $V = p$ and $W = q$, or $V = q$ and $W = p$.
Therefore, if $Y = r$, there are 2 possible ways to assign the numbers.

Case 2: $Y = s$
We have $Y = s$ and $Z = t$.
This leaves V, W, X (which satisfy $V < W < X$) to be assigned to p, q, r.
There is no known relationship between p, q, r.
Therefore, there are 3 ways to assign one of V, W, X to p.
For each of these 3 ways, there are 2 ways of assigning one of the two remaining numbers to q.
For each of these 3×2 ways, there is only 1 choice for the number assigned to r.
Overall, this gives $3 \times 2 \times 1 = 6$ ways to do this assignment. (The 6 ways to assign V, W, X to p, q, r, respectively, are VWX, VXW, WVX, WXV, XVW, XWV.)
Therefore, if $Y = s$, there are 6 possible ways to assign the numbers.

Having examined the two possibilities, there are $2 + 6 = 8$ different ways to assign the numbers.

Answer: (D)

24. Let x be the total number of students at Pascal H.S.
Let a be the total number of students who went on both the first trip and the second trip, but did not go on the third trip.
Let b be the total number of students who went on both the first trip and the third trip, but did not go on the second trip.
Let c be the total number of students who went on both the second trip and the third trip, but did not go on the first trip.
We note that no student went on one trip only, and that 160 students went on all three trips.
We draw a Venn diagram:



Since the total number of students at the school is x and each region in the diagram is labelled separately, then

$$x = a + b + c + 160$$

From the given information:

- 50% of the students in the school went on the first trip, so $0.5x = a + b + 160$
- 80% of the students in the school went on the second trip, so $0.8x = a + c + 160$
- 90% of the students in the school went on the third trip, so $0.9x = b + c + 160$

Combining all of this information,

$$\begin{aligned} x &= a + b + c + 160 \\ 2x &= 2a + 2b + 2c + 160 + 160 \\ 2x &= (a + b + 160) + (a + c + 160) + (b + c) \\ 2x &= 0.5x + 0.8x + (0.9x - 160) \\ 2x &= 2.2x - 160 \\ 160 &= 0.2x \\ x &= 800 \end{aligned}$$

Therefore, there are 800 students at Pascal High School.

Answer: (D)

25. We refer to the two sequences as the GEB sequence and the difference sequence.
Since the GEB sequence is increasing and since each positive integer that does not occur in the GEB sequence must occur in the difference sequence, then each positive integer less than 12 except 1, 3, 7 (a total of 8 positive integers) must occur in the difference sequence.
Since the difference sequence is increasing, then these 8 positive integers occur in increasing order.
Therefore, the difference sequence begins $2, 4, 5, 6, 8, 9, 10, 11, \ldots$.
This allows us to continue the GEB sequence using the integers in the difference sequence as the new differences between consecutive terms. For example, since the fourth term in the GEB sequence is 12 and the fourth difference from the difference sequence is 6, then the fifth term in the GEB sequence is $12 + 6 = 18$.
Continuing in this way, we can write out more terms in the GEB sequence:

$$1, 3, 7, 12, 18, 26, 35, 45, 56, \ldots$$

In a similar way, each positive integer less than 26 except 1, 3, 7, 12, 18 (a total of 20 positive integers) must occur in the difference sequence.
Since the difference sequence is increasing, then these 20 positive integers occur in increasing order.
Therefore, the difference sequence begins

$$2, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, \ldots$$

As above, we can write out more terms in the GEB sequence:

$$1, 3, 7, 12, 18, 26, 35, 45, 56, 69, 83, 98, 114, 131, \ldots$$

Again, every positive integer less than 114, with the exception of the 12 integers before 114 in the GEB sequence, must occur in the difference sequence, and these integers ($113 - 12 = 101$ of them in all) must occur in increasing order.
We need to determine the 100th term in the GEB sequence. We can do this by taking the first term in the GEB sequence (that is, 1) and adding to it the first 99 terms in the difference sequence. This is because the terms in the difference sequence are the differences between consecutive terms in the GEB sequence, so adding these to the first term allows us to move along the sequence.
From above, we see that 113 is 101st term in the difference sequence, so 112 is the 100th term, and 111 is the 99th term.
Since the first 99 terms in the difference sequence consist of most of the integers from 2 to 111, with the exception of a few (those in the GEB sequence), we can find the sum of these terms by adding all of the integers from 2 to 111 and subtracting the relevant integers.
Therefore, the 100th term in the GEB sequence equals

$$\begin{aligned}
&1 + (2 + 4 + 5 + 6 + 8 + \cdots + 109 + 110 + 111) \\
&\quad = 1 + (1 + 2 + 3 + 4 + \cdots + 109 + 110 + 111) \\
&\qquad\qquad -(1 + 3 + 7 + 12 + 18 + 26 + 35 + 45 + 56 + 69 + 83 + 98) \\
&\quad = 1 + \tfrac{1}{2}(111)(112) - (453) \\
&\qquad\qquad \text{(using the fact that the sum of the first } n \text{ positive integers is } \tfrac{1}{2}n(n+1)) \\
&\quad = 1 + 111(56) - 453 \\
&\quad = 5764
\end{aligned}$$

Thus, the 100th term in the GEB sequence is 5764.

Answer: (E)