The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2013 Fryer Contest

Thursday, April 18, 2013
(in North America and South America)

Friday, April 19, 2013
(outside of North America and South America)

Solutions

1. (a) Ann's average score for these two games was $\frac{103+117}{2} = \frac{220}{2}$ or 110.

   (b) *Solution 1*

   Since the average score of Bill's three games was 115, then the sum of his three scores was $115 \times 3 = 345$.
   The sum of Bill's scores in his first two games was $108 + 125 = 233$.
   Therefore, Bill's score in his third game was $345 - 233 = 112$.

   Solution 2

   Let x represent Bill's score in the third game.
   Since the average of the three games was 115, then $\dfrac{108 + 125 + x}{3} = 115$.
   Solving for x, we get $108 + 125 + x = 115 \times 3$ or $233 + x = 345$, so $x = 345 - 233 = 112$.
   Therefore, Bill's score in his third game was 112.

   (c) *Solution 1*

   Since the average score of Cathy's first three games was 113, then the sum of her first three scores was $113 \times 3 = 339$.
   If the average score of Cathy's first five games was 120, then the sum of her five scores was $120 \times 5 = 600$.
   Therefore, the sum of Cathy's scores in games four and five was $600 - 339 = 261$.
   Since Cathy scored the same in her fourth and fifth games, then each of these scores would equal $\frac{261}{2} = 130.5$.
   However, in bowling, each score is a whole number so this is not possible.
   Therefore, it is not possible for Cathy to have an average score of 120 in these five games.

   Solution 2

   Since the average score of Cathy's first three games was 113, then the sum of her first three scores was $113 \times 3 = 339$.
   Let y represent Cathy's score in game four and her score in game five (since they were the same score).
   Since the average of the five games was 120, then $\dfrac{339 + y + y}{5} = 120$.
   Solving for y, we get $339 + 2y = 120 \times 5$ or $339 + 2y = 600$, so $2y = 261$ and $y = 130.5$.
   However, in bowling, each score is a whole number so this is not possible.
   Therefore, it is not possible for Cathy to have an average score of 120 in these five games.

2. (a) The outside of the field consists of two straight sides each of length 100 m joined by two semi-circular arcs each of diameter 60 m.
   Thus, the perimeter of the field is equal to the total length of the two straight sides, or 200 m, added to the circumference of a circle with diameter 60 m, or $\pi(60)$ m.
   Thus, the perimeter of the field is $(200 + 60\pi)$ m.

   (b) Amy may run along the perimeter of the field from C to D in two possible ways.
   No matter which way Amy chooses, she will travel along one of the two straight sides, and one of the two semi-circular arcs.
   That is, Amy will travel exactly one half of the total perimeter of the field, or $(100+30\pi)$ m, in travelling from C to D.
   Billais runs in a straight line distance from C to D.
   Since each straight 100 m side is perpendicular to each 60 m diameter, then CD is the hypotenuse of a right triangle with the other two sides having length 100 m and 60 m.

By the Pythagorean Theorem, $CD^2 = 100^2 + 60^2$ or $CD = \sqrt{13\,600}$ m (since $CD > 0$).
Thus, Amy travels $100 + 30\pi - \sqrt{13\,600} \approx 78$ m farther than Billais.

(c) The outside of the track consists of two straight sides each of length 100 m joined by two semi-circular arcs each of diameter $60 + x + x$ or $(60 + 2x)$ m.
Thus, the perimeter of the outside of the track is equal to the total length of the two straight sides, or 200 m, added to the circumference of a circle with diameter $(60+2x)$ m, or $\pi(60 + 2x)$ m.
Thus, the perimeter of the outside of the track is $(200 + (60 + 2x)\pi)$ m.
Since the perimeter of the outside of the track is 450 m, then $200 + (60 + 2x)\pi = 450$.
Solving this equation for x, we get $(60 + 2x)\pi = 250$ or $60 + 2x = \frac{250}{\pi}$ or $2x = \frac{250}{\pi} - 60$ or $x = \frac{1}{2}(\frac{250}{\pi} - 60)$, and so $x = \frac{125}{\pi} - 30 \approx 9.789$.
Rounded to the nearest whole number, the value of x is 10.

3. (a) The four-digit number $51A3$ is divisible by 3 when the sum of its digits is divisible by 3.
The sum of its digits is $5 + 1 + A + 3 = 9 + A$.
The values for the digit A such that $9 + A$ is divisible by 3 are $0, 3, 6$, or 9.

(b) The four-digit number $742B$ is divisible by 2 when it is even.
Thus, $742B$ is divisible by 2 if B is $0, 2, 4, 6$, or 8.
The four-digit number $742B$ must also be divisible by 3.
The number $742B$ is divisible by 3 when $7 + 4 + 2 + B$ or $13 + B$ is divisible by 3.
Thus, $742B$ is divisible by 3 if B is $2, 5$ or 8.
The only values of the digit B such that the four-digit number $742B$ is divisible by 6 (divisible by both 2 and 3), are $B = 2$ or $B = 8$.

(c) An integer is divisible by 15 when it is divisible by both 5 and 3 (since the product of 5 and 3 is 15 and since 5 and 3 have no common factor).
The integer $1234PQPQ$ is divisible by 5 when its units digit, Q, is 0 or 5.
The integer $1234PQPQ$ is divisible by 3 when the sum of its digits, $1 + 2 + 3 + 4 + P + Q + P + Q$ or $10 + 2P + 2Q$, is divisible by 3.
We proceed by checking the two cases, $Q = 0$ and $Q = 5$.
When $Q = 0$, the sum of the digits $10 + 2P + 2Q$ is $10 + 2P$.
Therefore, when $Q = 0$ the values of P for which $10 + 2P$ is divisible by 3 are $1, 4$ and 7.
When $Q = 5$, the sum of the digits $10 + 2P + 2Q$ is $10 + 2P + 10$ or $20 + 2P$.
Therefore when $Q = 5$, the values of P for which $20 + 2P$ is divisible by 3 are $2, 5$ and 8.
The values of digits P and Q (written as (P, Q)) for which $1234PQPQ$ is divisible by 15 are $(1, 0), (4, 0), (7, 0), (2, 5), (5, 5)$, or $(8, 5)$.

(d) An integer is divisible by 12 when it is divisible by both 4 and 3 (since the product of 4 and 3 is 12 and since 4 and 3 have no common factor).
In the product $2CC \times 3D5$, $3D5$ is odd and thus cannot be divisible by 4.
Therefore, $2CC$ must be divisible by 4 and so must also be divisible by 2.
That is, C must be an even number. We proceed by checking $C = 0, 2, 4, 6, 8$.
If $C = 0$, 200 is divisible by 4 and so $C = 0$ is a possibility.
If $C = 2$, 222 is not divisible by 4 and so $C = 2$ is not a possibility.
If $C = 4$, 244 is divisible by 4 and so $C = 4$ is a possibility.
If $C = 6$, 266 is not divisible by 4 and so $C = 6$ is not a possibility.
Finally, if $C = 8$, 288 is divisible by 4 and so $C = 8$ is a possibility.
We now proceed by using these three cases, $C = 0$, $C = 4$ and $C = 8$, to determine possible values for D.

When $C = 0$, the product $2CC \times 3D5$ becomes $200 \times 3D5$.

The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3.
Therefore, $3D5$ must be divisible by 3 and so $8 + D$ is divisible by 3.
The possible values for D such that $8 + D$ is divisible by 3 are $1, 4$ and 7.
In this case, there are 3 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

When $C = 4$, the product $2CC \times 3D5$ becomes $244 \times 3D5$.
The number 244 is not divisible by 3 since the sum of its digits, 10, is not divisible by 3.
Therefore, $3D5$ must be divisible by 3 and as was shown in the first case, there are 3 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

When $C = 8$, the product $2CC \times 3D5$ becomes $288 \times 3D5$.
The number 288 is divisible by 3 since the sum of its digits, 18, is divisible by 3.
Therefore, the product $288 \times 3D5$ is divisible by 3 independent of the value of D.
That is, D can be any digit from 0 to 9.
In this case, there are 10 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

The total number of pairs of digits C and D such that $2CC \times 3D5$ is divisible by 12 is $3 + 3 + 10 = 16$.

4. (a) Since the dot starts at $(0, 0)$ and ends at $(1, 0)$, it moves a total of 1 unit right and 0 units up or down.
Since the dot moves 0 units up or down, then the number of $\uparrow$ moves equals the number of $\downarrow$ moves.
Since the total number of moves is at most 4, then the moves include 0 $\uparrow$ and 0 $\downarrow$ (0 moves in total), or 1 $\uparrow$ and 1 $\downarrow$ (2 moves in total), or 2 $\uparrow$ and 2 $\downarrow$ (4 moves in total).
Since the dot moves 1 unit right, then the number of $\rightarrow$ moves is 1 more than the number of $\leftarrow$ moves.
Since the total number of moves is at most 4, then there are 1 $\rightarrow$ and 0 $\leftarrow$ (1 move in total), or 2 $\rightarrow$ and 1 $\leftarrow$ (3 moves in total).
If there are 0 up/down moves, there can be 1 or 3 left/right moves.
If there are 2 up/down moves, there can be 1 left/right move.
If there are 4 up/down moves, there can be 0 left/right moves, which isn't possible.
With 0 up/down and 1 left/right moves, the sequence of moves is $\rightarrow$.
With 0 up/down and 3 left/right moves, the sequence of moves is $\leftarrow\rightarrow\rightarrow$ or any arrangement of these moves.
With 2 up/down and 1 left/right moves, the sequence of moves is $\uparrow\downarrow\rightarrow$ or any arrangement of these moves.
There are 3 possible arrangements of the moves $\leftarrow\rightarrow\rightarrow$. (This is because the $\leftarrow$ can be the first, second or third moves, and the two remaining moves are both $\rightarrow$.)
There are 6 possible arrangements of the moves $\uparrow\downarrow\rightarrow$. (This is because there are three moves that can go first, and then two remaining moves that can go second, and then one move that can go third, and so $3 \times 2 \times 1 = 6$ combinations of moves.)
Therefore, there are $1 + 3 + 6 = 10$ possible ways for the dot to end at $(1, 0)$.

(b) If the dot makes exactly 4 moves, these moves can consist of

- any combination of 4 right/left moves, or
- any combination of 3 right/left moves and any 1 up/down move, or
- any combination of 2 right/left moves and any combination of 2 up/down moves, or

- any 1 right/left move and any combination of 3 up/down moves, or
- any combination of 4 up/down moves.

Given a specific set of moves, any arrangement of these moves will move the dot to the same point, so we can think, for example, of all of the right/left moves occurring first followed by the up/down moves.
Four right/left moves can be 4 right (ending at $x = 4$), 3 right and 1 left (ending at $x = 3 - 1 = 2$), 2 right and 2 left (ending at $x = 0$), 1 right and 3 left (ending at $x = 1 - 3 = -2$), or 4 left (ending at $x = -4$).
Three right/left moves can be 3 right, 2 right and 1 left, 1 right and 2 left, or 3 left, ending at $x = 3, 1, -1, -3$, respectively.
Two right/left moves can be 2 right, 1 right and 1 left, or 2 left, ending at $x = 2, 0, -2$, respectively.
One right/left move can be 1 right or 1 left, ending at $x = 1, -1$, respectively.
No moves right or left will end at $x = 0$.
Similarly, four up/down moves can end at $y = 4, 2, 0, -2, -4$, three up/down moves can end at $y = 3, 1, -1, -3$, two up/down moves can end at $y = 2, 0, -2$, one up/down move can end at $y = 1, -1$, and no moves up or down will end at $y = 0$. (To see this, we can repeat the previous argument, replacing "right/left" with "up/down" and changing x-coordinates to y-coordinates.)
Given the total number of moves in each of the horizontal and vertical direction, any combination of the resulting x- and y-coordinates is possible, since the horizontal and vertical moves do not affect each other.
Therefore, if the dot makes exactly 4 moves, we have

- any combination of 4 right/left moves and no moves up or down ($x = 4, 2, 0, -2, -4$, and $y = 0$), ending at the points $(4, 0), (2, 0), (0, 0), (-2, 0), (-4, 0)$, or
- any combination of 3 right/left moves and any 1 up/down move ($x = 3, 1, -1, -3$, and $y = 1, -1$), ending at the points $(3, 1), (3, -1), (1, 1), (1, -1), (-1, 1), (-1, -1), (-3, 1)$, $(-3, -1)$, or
- any combination of 2 right/left moves and any combination of 2 up/down moves ($x = 2, 0, -2$, and $y = 2, 0, -2$), ending at the points $(2, 2), (2, 0), (2, -2), (0, 2), (0, 0)$, $(0, -2), (-2, 2), (-2, 0), (-2, -2)$, or
- any 1 right/left move and any combination of 3 up/down moves ($x = 1, -1$, and $y = 3, 1, -1, -3$), ending at the points $(1, 3), (1, 1), (1, -1), (1, -3), (-1, 3), (-1, 1)$, $(-1, -1), (-1, -3)$, or
- any combination of 4 up/down moves and no moves right or left ($x = 0$, and $y = 4, 2, 0, -2, -4$), ending at the points $(0, 4), (0, 2), (0, 0), (0, -2), (0, -4)$.

After careful observation of the list of ending points above, we see that a number of points can be reached in more than one way.
In counting the total number of distinct points we must take care to count the same point more than once.
In the first bullet above, there are 5 distinct points.
In the second bullet, there are 8 new points that have not yet been counted.
In the third bullet there are 9 points, however, we have already counted the 3 points $(2, 0), (0, 0)$, and $(-2, 0)$.
In the fourth bullet there are 8 points, however, we have already counted the 4 points $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$.
In the final bullet there are 5 points, however, we have already counted the 3 points

$(0,2), (0,0)$, and $(0,-2)$.
Therefore, the total number of possible points is $5+8+(9-3)+(8-4)+(5-3)=25$.

(c) The dot can get to $(-7,12)$ in 19 moves: 7 left and 12 up.
We need at least 19 moves for the dot to get to $(-7,12)$, since we need at least 7 moves to the left and at least 12 moves up (with possibly more moves that cancel each other out).
The dot can also get to $(-7,12)$ in 21 moves: 7 left, 12 up, 1 left, 1 right.
The dot can also get to $(-7,12)$ in 23 moves: 7 left, 12 up, 2 left, 2 right.
In a similar way, the dot can get to $(-7,12)$ in $19+2m$ moves for any non-negative integer m: 7 left, 12 up, m left, m right.
As m ranges from 0 to 40, the expression $19+2m$ takes every odd integer value from 19 to 99.
This is 41 integer values.
We have shown that if $k<19$, then the dot cannot get to $(-7,12)$ in k moves, and if $k\geq 19$ and k is odd, then the dot can get to $(-7,12)$ in k moves.
Finally, we show that if $k\geq 19$ and k is even, then the dot cannot get to $(-7,12)$ in k moves. This will complete our solution, and give an answer of 41, as above.
Suppose that $k\geq 19$ is even and that the dot can get to $(-7,12)$ in k moves.
These k moves include h right/left moves and v up/down moves, with $k=h+v$.
Since k is even, then h and v are both even or both odd. (This is because Even + Even is Even, Odd + Odd is Even, and Odd + Even is Odd.)
Suppose that h is even. The h right/left moves include r right moves and l left moves, which gives $h=r+l$ and $r-l=-7$, since the dot ends with x-coordinate -7.
Since h is even and $h=r+l$, then r and l are both even or both odd.
But if r and l are both even or both odd, then $r-l$ must be even.
This disagrees with the fact that $r-l=-7$.
Therefore, h cannot be even, and so h is odd.
Since h is odd, then v is odd.
But the v up/down moves include u up moves and d down moves, which gives $v=u+d$ and $u-d=12$.
Since $v=u+d$ is odd, then u and d are odd and even or even and odd, which means that $u-d$ is odd, and so cannot be equal to 12.
Therefore, none of the possibilities work.
This means that we cannot get to $(-7,12)$ in an even number of moves.
Thus, there are 41 positive integers k with $k\leq 100$ for which the dot can reach $(-7,12)$ in k moves.

(d) Consider any point P at which the dot can end in 47 moves. The dot can also end at this point in 49 moves by adding, for example, a cancelling pair such as $\rightarrow\leftarrow$ onto the end of the sequence of 47 moves ending at P.
Therefore, each of the 2304 points at which the dot can end in 47 moves is a point at which the dot can end in 49 moves.
Any other point Q at which the dot can end in 49 moves cannot be arrived at in 47 moves.
Consider any sequence of moves that takes the dot to Q. It cannot include both a right and a left move, or both an up and a down move, otherwise these moves could be "cancelled" and removed, giving a sequence of 47 moves ending at Q.
Thus, a sequence of moves ending at Q includes h horizontal moves and v vertical moves (with $h+v=49$), where the horizontal moves are either all right or all left moves, and the vertical moves are either all up or all down moves.
If $h=0$, then $v=49$, so the dot can end at $(0,49)$ (49 moves up) or at $(0,-49)$ (49 moves

down).
Similarly, if $v = 0$, the dot can end at $(49, 0)$ or $(-49, 0)$.
If h and v are both not zero, then there are both horizontal and vertical moves.
We consider the case where there are h right moves and v up moves, with $h + v = 49$.
Since $h \geq 1$ and $v \geq 1$, then h can take any value from 1 to 48 giving values of v from 48 to 1, and points $(1, 48), (2, 47), \ldots, (47, 2), (48, 1)$.
This is exactly 48 distinct points.
Similarly, in each of the cases of right moves and down moves, left moves and up moves, and left moves and down moves, we obtain 48 distinct points.
Therefore, the number of points Q at which the dot can end in 49, but not 47 moves, is $2 + 2 + 4 \times 48 = 196$.
This gives a total of $2304 + 196 = 2500$ points at which the dot can end in exactly 49 moves.