The CENTRE for EDUCATION
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2013 Canadian Senior Mathematics Contest

Thursday, November 21, 2013
(in North America and South America)

Friday, November 22, 2013
(outside of North America and South America)

Solutions

Part A

1. *Solution 1*
Since $ABCD$ is a parallelogram, then AB and DC are parallel and equal in length.
Since A and B both have a y-coordinate of 3, then AB is horizontal.
Since A has coordinates $(2, 3)$ and B has coordinates $(7, 3)$, then AB has length $7 - 2 = 5$.
Thus, DC is horizontal and has length 5.
Since D has y-coordinate 7, then C has y-coordinate 7.
Since DC has length 5 and the x-coordinate of D is 3, then the x-coordinate of C is $3 + 5 = 8$.
Therefore, the coordinates of C are $(8, 7)$.

Solution 2
Since $ABCD$ is a parallelogram, then AD and BC are parallel and have the same length.
To move from $A(2, 3)$ to $D(3, 7)$, we move to the right 1 unit and we move up 4 units (since the difference in x-coordinates is 1 and the difference in y-coordinates is 4).
Therefore, to get from $B(7, 3)$ to C, we also move 1 unit right and 4 units up.
Thus, the coordinates of C are $(7 + 1, 3 + 4) = (8, 7)$.

ANSWER: $(8, 7)$

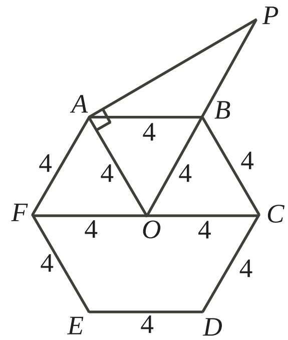
2. Since Ben is not given the number 1, he is given the number 2, 3 or 4.
Since Wendy's number and Riley's number are one apart, then their numbers are consecutive.
Note that Wendy's number is 1 greater than Riley's number.
If Ben is given 2, then Riley and Wendy must be given 3 and 4. In this case, Sara gets 1.
If Ben is given 3, then Riley and Wendy must be given 1 and 2. In this case, Sara gets 4.
If Ben is given 4, then Riley and Wendy must be given 1 and 2, or 2 and 3. In the first of these cases, Sara must be given 3. In the second of these cases, Sara must be given 1.
Therefore, Sara can be given 1, 3 and 4, and so cannot be given the number 2.

ANSWER: 2

3. Note that $99! = 99(98)(97)\cdots(3)(2)(1)$ and $101! = 101(100)(99)(98)(97)\cdots(3)(2)(1)$.
Thus, $101! = 101(100)(99!)$.
Therefore,
$$\frac{99!}{101! - 99!} = \frac{99!}{101(100)(99!) - 99!} = \frac{99!}{99!(101(100) - 1)} = \frac{1}{101(100) - 1} = \frac{1}{10\,099}$$
Therefore, $n = 10\,099$.

ANSWER: $n = 10\,099$

4. Join FO and OC.
Since $ABCDEF$ is a regular hexagon with side length 4, then $FA = AB = BC = 4$.
Since $ABCDEF$ is a regular hexagon with centre O, then $FO = AO = BO = CO$.
Therefore, $\triangle FOA$, $\triangle AOB$ and $\triangle BOC$ are congruent triangles, by symmetry, since their corresponding side lengths are equal.
Also, each of $\angle FOA$, $\angle AOB$ and $\angle BOC$ must be $\frac{1}{6}$ of the total angle around O.
Thus, $\angle FOA = \angle AOB = \angle BOC = \frac{1}{6}(360^\circ) = 60^\circ$.
Since $\triangle AOB$ is isosceles with $AO = BO$ and $\angle AOB = 60^\circ$, then $\angle OAB = \angle OBA$, so $\angle OAB = \frac{1}{2}(180^\circ - \angle AOB) = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
Thus, $\triangle AOB$ is equilateral, and so $AO = AB = 4$.



Now $\triangle OAP$ has $\angle OAP = 90^\circ$ and $\angle AOP = \angle AOB = 60^\circ$. This means that $\triangle OAP$ is a 30°-60°-90° triangle.
By the ratio of sides in a 30°-60°-90°, $AP = \sqrt{3}(AO) = 4\sqrt{3}$.
Therefore, the area of $\triangle OAP$ is $\frac{1}{2}(OA)(AP) = \frac{1}{2}(4)(4\sqrt{3}) = 8\sqrt{3}$.

Answer: $8\sqrt{3}$

5. We recall that a positive integer is divisible by 3 whenever the sum of its digits is divisible by 3. Since the sum of the digits does not depend on the order of the digits, then rearranging the digits of a positive integer that is divisible by 3 produces another positive integer that is divisible by 3.
Note that 10 000 is not divisible by 3. Every other positive integer between 1000 and 10 000 is a four-digit integer.
Consider a four-digit positive integer whose four digits are consecutive integers. We can rearrange the digits of this integer in decreasing order to obtain one of the positive integers 3210, 4321, 5432, 6543, 7654, 8765, and 9876.
The sums of the digits of these integers are 6, 10, 14, 18, 22, 26, and 30, respectively.
Of these, 6, 18 and 30 are the only sums that are divisible by 3, so 3210, 6543 and 9876 are the only ones divisible by 3.
Since rearranging the digits does not affect whether an integer is divisible by 3, then a four-digit integer satisfies the given conditions if its digits are rearrangements of 3210 or 6543 or 9876.
There are 24 four-digit integers whose digits are rearrangements of 6543. (There are 4 possibilities for the thousands digit, then 3 possibilities for the hundreds digit, then 2 possibilities for the tens digit, and 1 possibility for the units digit, and so $4 \times 3 \times 2 \times 1 = 24$ integers that use these digits.)
Similarly, there are 24 four-digit integers whose digits are rearrangements of 9876.
Finally, there are 18 four-digit integers whose digits are rearrangements of 3210. (There are 3 possibilities for the thousands digits (because 0 cannot be the thousands digit), then 3 possibilities for the hundreds digit, 2 for the tens digit, and 1 for the units digit, and so $3 \times 3 \times 2 \times 1 = 18$ integers that use these digits.)
In total, there are $24 + 24 + 18 = 66$ positive integers with these three properties.

Answer: 66

6. *Solution 1*

We consider two cases: $y < 60$ and $y \geq 60$.
We use the facts that $\min(a, b) \leq a$, $\min(a, b) \leq b$, $\max(a, b) \geq a$, and $\max(a, b) \geq b$. In other words, the minimum of two numbers is less than or equal to both numbers and the maximum of two number is greater than or equal to both numbers.
We also use the fact that if $a \leq b$, then $\min(a, b) = a$ and $\max(a, b) = b$. Note that the case of $a = b$ does not affect these equations.

<u>Case 1: $y < 60$</u>
If $y < 60$, then $\min(x, y) \leq y$ and $y < 60$, so $\min(x, y) < 60$.
Since $\min(x, y) < 60$, then $\text{LS} = \max(60, \min(x, y)) = 60$.
Also, $\max(60, x) \geq 60$, so since $y < 60$, then $\text{RS} = \min(\max(60, x), y) = y$.
Therefore, in the case $y < 60$, the equation is satisfied whenever $60 = y$, which is never true.
Therefore, there are no ordered pairs (x, y) with $y < 60$ that satisfy the equation.

<u>Case 2: $y \geq 60$</u>
If $x < y$, then $\min(x, y) = x$ and so $\text{LS} = \max(60, x)$.
If $x < y$, then both 60 and x are no larger than y, so the largest of 60 and x is no larger than y. In other words, $\max(60, x) \leq y$ and so $\text{RS} = \min(\max(60, x), y) = \max(60, x)$.
If $x < y$, the equation becomes $\max(60, x) = \max(60, x)$ and so is true for every ordered pair (x, y).
If $x \geq y$, then $\min(x, y) = y$ and so $\text{LS} = \max(60, y) = y$ since $y \geq 60$.
If $x \geq y$, then $x \geq 60$ so $\max(60, x) = x$ and so $\text{RS} = \min(x, y) = y$.
If $x \geq y$, the equation becomes $y = y$ and so is true for every pair (x, y).

Therefore, the given equation is satisfied by all pairs (x, y) with $y \geq 60$.
Since $1 \leq x \leq 100$, there are 100 possible values for x.
Since $60 \leq y \leq 100$, there are 41 possible values for y.
Therefore, there are $100 \times 41 = 4100$ ordered pairs that satisfy the equation.

Solution 2

In this solution, we use the facts that if $a \leq b$, then $\min(a, b) = a$ and $\max(a, b) = b$.
We also label as $(*)$ the original equation $\max(60, \min(x, y)) = \min(\max(60, x), y)$.

We consider the possible arrangements of the integers 60, x and y.
There are six possible arrangements. We examine the various pieces of the left and right sides of the given equation for each of these orders, moving from the innermost function out on each side:

Case	$\min(x, y)$	$\text{LS} = \max(60, \min(x, y))$	$\max(60, x)$	$\text{RS} = \min(\max(60, x), y)$
$60 \leq x \leq y$	$= x$	$= \max(60, x) = x$	$= x$	$= \min(x, y) = x$
$60 \leq y \leq x$	$= y$	$= \max(60, y) = y$	$= x$	$= \min(x, y) = y$
$x \leq 60 \leq y$	$= x$	$= \max(60, x) = 60$	$= 60$	$= \min(60, y) = 60$
$x \leq y \leq 60$	$= x$	$= \max(60, x) = 60$	$= 60$	$= \min(60, y) = y$
$y \leq 60 \leq x$	$= y$	$= \max(60, y) = 60$	$= x$	$= \min(x, y) = y$
$y \leq x \leq 60$	$= y$	$= \max(60, y) = 60$	$= 60$	$= \min(60, y) = y$

If $60 \leq x \leq y$, then $(*)$ is equivalent to $x = x$, so all (x, y) with $60 \leq x \leq y$ satisfy $(*)$.
If $60 \leq y \leq x$, then $(*)$ is equivalent to $y = y$, so all (x, y) with $60 \leq y \leq x$ satisfy $(*)$.
If $x \leq 60 \leq y$, then $(*)$ is equivalent to $60 = 60$, so all (x, y) with $x \leq 60 \leq y$ satisfy $(*)$.
If $x \leq y \leq 60$, then $(*)$ is equivalent to $60 = y$, so only (x, y) with $y = 60$ and $x \leq 60$ satisfy

$(*)$. These (x, y) are already accounted for in the third case.
If $y \leq 60 \leq x$, then $(*)$ is equivalent to $60 = y$, so only (x, y) with $y = 60$ and $x \geq 60$ satisfy $(*)$. These (x, y) are included in the second case.
If $y \leq x \leq 60$, then $(*)$ is equivalent to $60 = y$, so only $y = 60$ and $60 \leq x \leq 60$, which implies $x = 60$. In this case, only the pair $(60, 60)$ satisfies $(*)$ and this pair is already included in the first case.

Therefore, we need to count the pairs of positive integers (x, y) with $x \leq 100$ and $y \leq 100$ which satisfy one of $60 \leq x \leq y$ or $60 \leq y \leq x$ or $x \leq 60 \leq y$.
From these three ranges for x and y we see that $y \geq 60$ in all cases and x can either be less than or equal to 60 or can be greater than or equal to 60 and either larger or smaller than y.
In other words, these three ranges are equivalent to the condition that $y \geq 60$.
Therefore, $(*)$ is satisfied by all pairs (x, y) with $y \geq 60$.
Since $1 \leq x \leq 100$, there are 100 possible values for x.
Since $60 \leq y \leq 100$, there are 41 possible values for y.
Therefore, there are $100 \times 41 = 4100$ ordered pairs that satisfy the equation.

Answer: 4100

Part B

1. (a) In the second bank of lockers, each of the first two columns consists of two lockers.
The first column consists of lockers 21 and 22, and the second column consists of lockers 23 and 24.
Thus, the sum of the locker numbers in the column containing locker 24 is $23 + 24 = 47$.

(b) Suppose that a column contains two lockers, numbered x and $x + 1$, for some positive integer x.
Then the sum of the locker numbers in this column is $x + (x + 1) = 2x + 1$, which is odd.
Suppose that a column contains four lockers, numbered y, $y+1$, $y+2$, and $y+3$, for some positive integer y.
Then the sum of the locker numbers in this column is

$$y + (y + 1) + (y + 2) + (y + 3) = 4y + 6 = 2(2y + 3)$$

which is even.
Since 123 is odd, then there must be two lockers in the column which has locker numbers that add to 123.
If these locker numbers are x and $x + 1$, then we need to solve $2x + 1 = 123$, which gives $2x = 122$ or $x = 61$.
Therefore, the locker numbers in this column are 61 and 62.
(Note that since there are 20 lockers in each bank, then locker 61 is the first locker in the fourth bank, so does start a column containing two lockers.)

(c) Since 538 is even, then from (b), it must be the sum of four locker numbers.
If these numbers are y, $y + 1$, $y + 2$, and $y + 3$, then we need to solve $4y + 6 = 538$, which gives $4y = 532$ or $y = 133$.
Therefore, the locker numbers are $133, 134, 135, 136$.
(Note that these four numbers do appear together in one column, since locker 140 ends the seventh bank of lockers, so lockers 137, 138, 139, 140 form the last column, giving the four lockers 133, 134, 135, 136 as the previous column.)

(d) Since 2013 is odd, then if it were the sum of locker numbers in a column, it would be the sum of two locker numbers, as we saw in (b).
If these two locker numbers are x and $x + 1$, then $2x + 1 = 2013$ or $2x = 2012$, and so $x = 1006$.
This would mean that the locker numbers were 1006 and 1007.
But the first locker number in each column is odd. This is because there is an even number of lockers in each column and so the last locker number in each column is even, making the first locker number in the next column odd.
Therefore, lockers 1006 and 1007 do not form a column of two lockers, so there is no column whose locker numbers have a sum of 2013.

2. (a) Expanding and simplifying, we obtain

$$(a - 1)(6a^2 - a - 1) = 6a^3 - a^2 - a - 6a^2 + a + 1 = 6a^3 - 7a^2 + 1$$

(b) To solve the equation $6\cos^3\theta - 7\cos^2\theta + 1 = 0$, we first make the substitution $a = \cos\theta$.
The equation becomes $6a^3 - 7a^2 + 1 = 0$.
From (a), factoring the left side gives the equation $(a - 1)(6a^2 - a - 1) = 0$.

We further factor $6a^2 - a - 1$ as $(3a+1)(2a-1)$.
This gives $(a-1)(3a+1)(2a-1) = 0$.
Thus, $a = 1$ or $a = -\frac{1}{3}$ or $a = \frac{1}{2}$.
This tells us that the solutions to the original equation are the values of θ in the range $-180^\circ < \theta < 180^\circ$ with $\cos\theta = 1$ or $\cos\theta = -\frac{1}{3}$ or $\cos\theta = \frac{1}{2}$.
If $-180^\circ < \theta < 180^\circ$ and $\cos\theta = 1$, then $\theta = 0^\circ$.
If $-180^\circ < \theta < 180^\circ$ and $\cos\theta = -\frac{1}{3}$, then $\theta \approx 109.5^\circ$ or $\theta \approx -109.5^\circ$. (The positive value for θ can be obtained using a calculator. The negative value can be obtained by thinking about $\cos\theta$ as an even function of θ or by picturing either the graph of $y = \cos\theta$ or the unit circle.)
If $-180^\circ < \theta < 180^\circ$ and $\cos\theta = \frac{1}{2}$, then $\theta = 60^\circ$ or $\theta = -60^\circ$.
Therefore, the solutions to the equation $6\cos^3\theta - 7\cos^2\theta + 1 = 0$, rounded to one decimal place as appropriate, are $0^\circ, 60^\circ, -60^\circ, 109.5^\circ, -109.5^\circ$.

(c) To solve the inequality $6\cos^3\theta - 7\cos^2\theta + 1 < 0$, we factor the left side to obtain

$$(\cos\theta - 1)(3\cos\theta + 1)(2\cos\theta - 1) < 0$$

From (b), we know the values of θ at which the left side equals 0, so we examine the intervals between these values and look at the sign (positive or negative) of each of the factors in these intervals. We complete the final column of the table below by noting that the product of three positive numbers is positive, the product of two positives with one negative is negative, the product of one positive and two negatives is positive, and the product of three negatives is negative.

Range of θ	Range of $\cos\theta$	$\cos\theta - 1$	$3\cos\theta + 1$	$2\cos\theta - 1$	Product
$-180^\circ < \theta < -109.5^\circ$	$-1 < \cos\theta < -\frac{1}{3}$	$-$	$-$	$-$	$-$
$-109.5^\circ < \theta < -60^\circ$	$-\frac{1}{3} < \cos\theta < \frac{1}{2}$	$-$	$+$	$-$	$+$
$-60^\circ < \theta < 0^\circ$	$\frac{1}{2} < \cos\theta < 1$	$-$	$+$	$+$	$-$
$0^\circ < \theta < 60^\circ$	$\frac{1}{2} < \cos\theta < 1$	$-$	$+$	$+$	$-$
$60^\circ < \theta < 109.5^\circ$	$-\frac{1}{3} < \cos\theta < \frac{1}{2}$	$-$	$+$	$-$	$+$
$109.5^\circ < \theta < 180^\circ$	$-1 < \cos\theta < -\frac{1}{3}$	$-$	$-$	$-$	$-$

From this analysis, the values of θ for which $6\cos^3\theta - 7\cos^2\theta + 1 < 0$ are

$$-180^\circ < \theta < -109.5^\circ \quad \text{and} \quad -60^\circ < \theta < 0^\circ \quad \text{and} \quad 0^\circ < \theta < 60^\circ \quad \text{and} \quad 109.5^\circ < \theta < 180^\circ$$

We could also have determined these intervals by looking only at positive values for θ and then using the fact that $\cos\theta$ is an even function to determine the negative values.

3. (a) A $(2,2)$-sequence obeys the rules that if $x_i = A$, then $x_{i+2} = B$ and if $x_i = B$, then $x_{i+2} = A$.
Suppose that a $(2,2)$-sequence has $x_1 = A$.
Then $x_{1+2} = x_3 = B$ and $x_{3+2} = x_5 = A$ and $x_7 = B$ and $x_9 = A$ and so on.
Following this pattern, every odd-numbered term in the sequence is determined by $x_1 = A$ and these terms alternate $A, B, A, B, \ldots$.
Similarly, suppose that a $(2,2)$-sequence has $x_1 = B$.
Then $x_{1+2} = x_3 = A$ and $x_{3+2} = x_5 = B$ and $x_7 = A$ and $x_9 = B$ and so on.
Following this pattern, every odd-numbered term in the sequence is determined by $x_1 = B$

and these terms alternate $B, A, B, A, \ldots$.
Note that the value of x_1 does not affect any of the even-numbered terms.
Therefore, the value of x_1 determines all of the odd-numbered terms in the sequence.
If a $(2,2)$-sequence has $x_2 = A$, then we will have $x_4 = B$, $x_6 = A$, $x_8 = B$, and so on, and if a $(2,2)$-sequence has $x_2 = B$, then we will have $x_4 = A$, $x_6 = B$, $x_8 = A$, and so on.
Therefore, the value of x_2 determines all of the even-numbered terms in the sequence.
There are 2 possible values for x_1.
There are 2 possible values for x_2.
Thus, there are $2 \times 2 = 4$ possible $(2,2)$-sequences.
These are

$$AABBAABBAA\ldots \qquad ABBAABBAAB\ldots$$

$$BAABBAABBA\ldots \qquad BBAABBAABB\ldots$$

(b) A $(1,2)$-sequence obeys the rules that if $x_i = A$, then $x_{i+1} = B$ and if $x_i = B$, then $x_{i+2} = A$.
There are only two possibilities: $x_1 = A$ or $x_1 = B$.

Suppose that a $(1,2)$-sequence exists with $x_1 = A$.
Then $x_{1+1} = x_2 = B$ and $x_{2+2} = x_4 = A$ and $x_{4+1} = x_5 = B$.
So x_1, x_2, x_3, x_4, x_5 is A, B, x_3, A, B.
Consider x_3. If $x_3 = B$, then we would have $x_5 = A$, which is not true. If $x_3 = A$, then we would have $x_4 = B$, which is not true.
Since there is no possible value for x_3, then a $(1,2)$-sequence cannot have $x_1 = A$.

Suppose that a $(1,2)$-sequence exists with $x_1 = B$.
Then $x_3 = A$ and $x_4 = B$.
So x_1, x_2, x_3, x_4 is B, x_2, A, B.
Consider x_2. If $x_2 = B$, then we would have $x_4 = A$, which is not true. If $x_2 = A$, then we would have $x_3 = B$, which is not true.
Since there is no possible value for x_2, then a $(1,2)$-sequence cannot have $x_1 = B$.

Therefore, a $(1,2)$-sequence cannot have $x_1 = A$ or $x_1 = B$, so no $(1,2)$-sequence exists.

(c) Suppose that $x_1, x_2, x_3, \ldots$ is an (m,n)-sequence.
Consider the sequence $y_1, y_2, y_3, \ldots$ defined by

$$y_1 = y_2 = \cdots = y_r = x_1, y_{r+1} = y_{r+2} = \cdots = y_{2r} = x_2, \ldots$$

In general, we define $y_{(q-1)r+1} = y_{(q-1)r+2} = \cdots = y_{qr} = x_q$ for each positive integer q.
In other words, the first r terms of the sequence equal x_1, the next r terms equal x_2, the next r terms equal x_3, and so on, with the qth group of r terms equal to x_q:

$$\overbrace{x_1, x_1, \ldots, x_1}^{r \text{ times}}, \overbrace{x_2, x_2, \ldots, x_2}^{r \text{ times}}, \ldots, \overbrace{x_q, x_q, \ldots, x_q}^{r \text{ times}}, \ldots$$

For example, consider the $(2,2)$-sequence $ABBAABBAABBA\ldots$.
With $r = 3$, this process would form the sequence

$$AAABBBBBBAAAAAABBBBBBAAAAAABBBBBBAAA\ldots$$

which we claim is a $(6,6)$-sequence.
We show that in general the sequence $y_1, y_2, y_3, \ldots$ is an (rm, rn)-sequence.
Consider a term y_i with $(q-1)r + 1 \le i \le qr$.

Then $y_i = x_q$.
(In the given example, consider the term $y_{11} = A$. Note that $3(3) + 1 \leq 11 \leq 4(3)$, so $y_{11} = x_4$.)
We must show that if $y_i = A$, then $y_{i+rm} = B$ and if $y_i = B$, then $y_{i+rn} = A$.
(Since $y_{11} = A$, we want to show that $y_{11+6} = B$.)
If $y_i = x_q = A$, then $x_{q+m} = B$ since the x's form an (m, n)-sequence.
(Since $x_4 = A$, then $x_6 = B$.)
Consider y_{i+rm}. Since $(q-1)r + 1 \leq i \leq qr$, then $(q-1)r + 1 + rm \leq i + rm \leq qr + rm$ or $(q + m - 1)r + 1 \leq i + rm \leq (q + m)r$.
By definition, $y_{i+rm} = x_{q+m}$. Since $x_{q+m} = B$, then $y_{i+rm} = B$ as required.
(We are looking at y_{17}. Since $5(3) + 1 \leq 17 \leq 6(3)$, then $y_{17} = x_6 = B$, as required.)
If $y_i = x_q = B$, then $x_{q+n} = A$ since the x's form an (m, n)-sequence.
Consider y_{i+rn}. Since $(q-1)r + 1 \leq i \leq qr$, then $(q-1)r + 1 + rn \leq i + rn \leq qr + rn$ or $(q + n - 1)r + 1 \leq i + rn \leq (q + n)r$.
By definition, $y_{i+rn} = x_{q+n}$. Since $x_{q+n} = A$, then $y_{i+rn} = A$ as required.
Therefore, the sequence $y_1, y_2, y_3, \ldots$ is an (rm, rn)-sequence.
Thus, if an (m, n)-sequence exists, then an (rm, rn)-sequence exists.

(d) Any positive integer m can be written in a unique way in the form $m = 2^p c$ where p is a non-negative integer and c is an odd integer. (To see this, we factor 2s out of m until the quotient is odd; this quotient is c and the number of 2s factored out is p.)
We write $m = 2^p c$ and $n = 2^q d$ where p and q are non-negative integers and c and d are odd positive integers.
We prove that an (m, n)-sequence exists if and only if m and n contain exactly the same number of factor of 2s (that is, if and only if $p = q$).
We proceed through a number of steps.

Step 1: A (c, d)-sequence exists whenever c and d are both odd positive integers
Consider the sequence $x_1, x_2, x_3, \ldots$ in which every odd-numbered term is A and every even-numbered term is B.
That is, the sequence is $ABABAB\ldots$.
We prove that this sequence is a (c, d)-sequence.
Suppose that $x_i = A$. Then i must be odd, since only odd-numbered terms equal A.
Since i is odd and c is odd, then $i + c$ is even, and so $x_{i+c} = B$.
Thus, if $x_i = A$, then $x_{i+c} = B$.
Suppose that $x_i = B$. Then i must be even, since only even-numbered terms equal B.
Since i is even and d is odd, then $i + d$ is odd, and so $x_{i+d} = A$.
Thus, if $x_i = B$, then $x_{i+d} = A$.
Therefore, $ABABAB\ldots$ is a (c, d)-sequence, so a (c, d)-sequence exists whenever c and d are both odd positive integers.

Step 2: An (m, n)-sequence exists if $m = 2^p c$ and $n = 2^p d$
Here, c and d are odd positive integers and p is a non-negative integer.
By Step 1, a (c, d)-sequence exists.
By (c) with $r = 2^p$, this implies that a $(2^p c, 2^p d)$-sequence exists, so an (m, n)-sequence exists.

We have shown that if m and n contain the same number of factors of 2, then an (m, n)-sequence exists. We must now show that if m and n do not contain the same number of factors of 2, then an (m, n)-sequence does not exist.

Step 3: If an (rm, rn)-sequence exists, then an (m, n)-sequence exists

Here, r, m, n are positive integers.
Suppose that $y_1, y_2, y_3, \ldots$ is an (rm, rn)-sequence.
Consider the sequence $x_1, x_2, x_3, \ldots$ where $x_i = y_{ri}$. (This is the sequence $y_r, y_{2r}, y_{3r}, \ldots$.)
We show that this new sequence is an (m, n)-sequence.
Suppose that $x_i = A$. Then $y_{ri} = A$ and so $y_{ri+rm} = B$.
But $ri + rm = r(i + m)$ and so $x_{i+m} = y_{ri+rm} = B$.
Suppose that $x_i = B$. Then $y_{ri} = B$ and so $y_{ri+rn} = A$.
But $ri + rn = r(i + n)$ and so $x_{i+n} = y_{ri+rn} = A$.
Therefore, $x_1, x_2, x_3, \ldots$ is an (m, n)-sequence as required. Thus, if an (rm, rn)-sequence exists, then an (m, n)-sequence exists.

Step 4: If an (m, n)-sequence exists, then an (n, m)-sequence exists

Here, m and n are positive integers.
Suppose that $x_1, x_2, x_3, \ldots$ is an (m, n)-sequence, and consider the sequence $y_1, y_2, y_3, \ldots$ defined by $y_i = B$ if $x_i = A$ and $y_i = A$ if $x_i = B$.
We show that $y_1, y_2, y_3, \ldots$ is an (n, m)-sequence.
If $y_i = A$, then $x_i = B$ and so $x_{i+n} = A$ which means that $y_{i+n} = B$.
Thus, whenever $y_i = A$, we have $y_{i+n} = B$.
If $y_i = B$, then $x_i = A$ and so $x_{i+m} = B$ which means that $y_{i+m} = A$.
Thus, whenever $y_i = B$, we have $y_{i+m} = A$.
This means that $y_1, y_2, y_3, \ldots$ is an (n, m)-sequence.
Therefore, whenever an (m, n)-sequence exists, then an (n, m)-sequence exists.
This also implies that if an (n, m)-sequence does not exist, then an (m, n)-sequence does not exist.

Step 5: Supplementing definition of (m, n)-sequence

Suppose that $x_1, x_2, x_3, \ldots$ is an (m, n)-sequence.
We know that if $x_i = A$, then $x_{i+m} = B$ and if $x_i = B$, then $x_{i+n} = A$.
Suppose that $x_i = A$. Then $x_{i+m} = B$ and further $x_{i+m+n} = A$.
What can we say about x_{i+n}? If $x_{i+n} = A$, then we would have $x_{i+m+n} = B$, which isn't the case.
Thus, if $x_i = A$, then $x_{i+n} = B$.
Similarly, we can show that if $x_i = B$, then $x_{i+m} = A$.
Therefore, in an (m, n)-sequence, we have that if $x_i = A$, then $x_{i+m} = B$ and $x_{i+n} = B$ and if $x_i = B$, then $x_{i+n} = A$ and $x_{i+m} = A$.

Step 6: If m is odd and n is even, an (m, n)-sequence does not exist

Suppose that $x_1, x_2, x_3, \ldots$ is an (m, n)-sequence.
Suppose that $x_1 = A$.
Consider the term x_{1+mn}.
First, we approach x_{1+mn} by considering every mth term starting at x_1.
From the definition and Step 5, we have $x_1 = A$, $x_{1+m} = B$, $x_{1+2m} = A$, $x_{1+3m} = B$, and so on.
Since n is even, then we move an even number of steps from x_1 to x_{1+mn} in this way, and so $x_{1+mn} = A$.
Next, we approach x_{1+mn} by considering every nth term starting at x_1.
From the definition and Step 5, we have $x_1 = A$, $x_{1+n} = B$, $x_{1+2n} = A$, $x_{1+3n} = B$, and so on.
Since m is odd, then we move an odd number of steps from x_1 to x_{1+mn} in this way, and

so $x_{1+mn} = B$.
This is a contradiction, and so we cannot have $x_1 = A$.
In a similar way, we can show that if $x_1 = B$, then the term x_{1+mn} leads to a contradiction.
Therefore, if m is odd and n is even, an (m, n)-sequence does not exist.
By Step 4, this also implies that if m is even and n is odd, an (m, n)-sequence does not exist.

Step 7: m and n do not contain the same number of factors of 2
We show that no (m, n)-sequence exists.
Suppose that $m = 2^p c$ and $n = 2^q d$ with p and q non-negative integers $p \neq q$ and c and d odd positive integers.
Suppose without loss of generality that $p < q$.
By Step 3, if a $(2^p c, 2^q d)$-sequence exists, then a $(c, 2^{q-p} d)$-sequence exists (using $r = 2^p$).
But c is odd and $2^{q-p} d$ is even, so by Step 6 such a sequence doesn't exist and so an (m, n)-sequence does not exist.
Therefore, if m and n do not contain the same number of 2s, then no (m, n)-sequence exists.

In conclusion, an (m, n)-sequence exists if and only if m and n contain the same number of factors of 2.