The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2012 Pascal Contest

(Grade 9)

Thursday, February 23, 2012
(in North America and South America)

Friday, February 24, 2012
(outside of North America and South America)

Solutions

1. Using the correct order of operations, $\dfrac{1+(3\times 5)}{2} = \dfrac{1+15}{2} = \dfrac{16}{2} = 8$.

   Answer: (D)

2. Of the 200 students, 50% (or one-half) of the students chose Greek food.
   Since one-half of 200 is 100, then 100 students chose Greek food.

   Answer: (E)

3. Since $\frac{60}{8} = 60 \div 8 = 7.5$, then this choice is not equal to a whole number.
   Note as well that $\frac{60}{12} = 5$, $\frac{60}{5} = 12$, $\frac{60}{4} = 15$, and $\frac{60}{3} = 20$ are all whole numbers.

   Answer: (B)

4. If 7:30 a.m. was 16 minutes ago, then it is currently $30 + 16 = 46$ minutes after 7:00 a.m., or 7:46 a.m.
   Since 8:00 a.m. is 60 minutes after 7:00 a.m., then it will be 8:00 a.m. in $60 - 46 = 14$ minutes.

   Answer: (B)

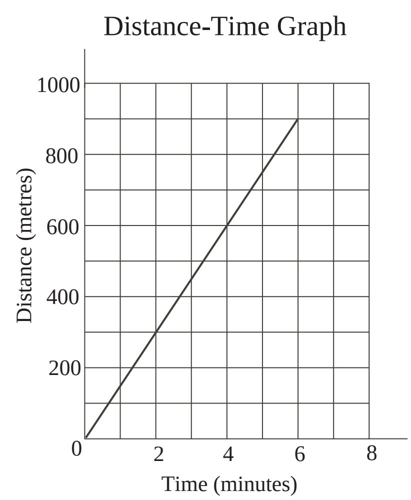
5. First, we write out the powers of 10 in full to obtain $8 \times 100\,000 + 4 \times 1000 + 9 \times 10 + 5$.
   Simplifying, we obtain $800\,000 + 4000 + 90 + 5$ or $804\,095$.

   Answer: (A)

6. We write the list in increasing order: $0.023, 0.032, 0.203, 0.302, 0.320$.
   The difference between the largest and smallest of these numbers is $0.320 - 0.023 = 0.297$.

   Answer: (E)

7. If Anna walked 600 metres in 4 minutes, then she walked $\frac{600}{4} = 150$ metres each minute.
   Therefore, in 6 minutes, she walked $6 \times 150 = 900$ metres.



   Answer: (D)

8. *Solution 1*
   The segment of the ruler between each integer marking is divided into 4 equal pieces (that is, is divided into quarters).
   Therefore, the point Q is at $2 + \frac{3}{4} = 2\frac{3}{4}$, and P is at $\frac{2}{4} = \frac{1}{2}$.
   Thus, the length of PQ is $2\frac{3}{4} - \frac{1}{2} = \frac{11}{4} - \frac{2}{4} = \frac{9}{4} = 2\frac{1}{4}$.
   Writing this as a decimal, we obtain 2.25.

   Solution 2
   The segment of the ruler between each integer marking is divided into 4 equal pieces (that is, is divided into quarters).
   There are 9 of these pieces between P and Q.
   Therefore, the length of PQ is $9 \times \frac{1}{4} = \frac{9}{4} = 2.25$.

   Answer: (A)

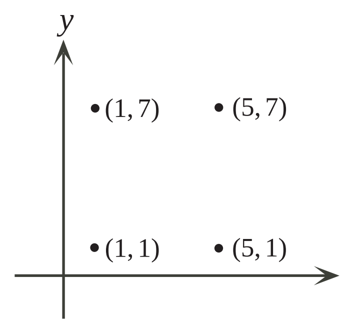
9. Substituting $y = 1$ into the second equation, we obtain $4x - 2(1) + 3 = 3x + 3(1)$.
Simplifying, we obtain $4x - 2 + 3 = 3x + 3$ or $4x + 1 = 3x + 3$.
Therefore, $4x - 3x = 3 - 1$ or $x = 2$.

Answer: (C)

10. Since Emily is a doctor and there are 5 doctors and 3 nurses aside from Emily at the hospital, then there are 6 doctors and 3 nurses in total.
Since Robert is a nurse, then aside from Robert, there are 6 doctors and 2 nurses.
Therefore, $d = 6$ and $n = 2$, so $dn = 12$.

Answer: (B)

11. Since the given three points already form a right angle, then the fourth vertex of the rectangle must be vertically above the point $(5, 1)$ and horizontally to the right of $(1, 7)$.
Therefore, the x-coordinate of the fourth vertex is 5 and the y-coordinate is 7.
Thus, the coordinates of the fourth vertex are $(5, 7)$.



Answer: (C)

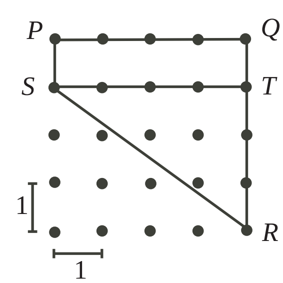
12. We can rephrase the given information by saying that each of the seven students paid \$3.71 and some of the students paid an extra \$0.01.
Since $7 \times \$3.71 = \$25.97$ and the pizza cost \$26.00 in total, then the students who paid the extra \$0.01 each must make up the final \$0.03 of the cost of the pizza.
Therefore, 3 students each paid an additional \$0.01 and so paid \$3.72 in total.

Answer: (B)

13. Using the definition of the operation, $g \nabla 6 = 45$ gives $g^2 - 6^2 = 45$.
Thus, $g^2 = 45 + 36 = 81$.
Since $g > 0$, then $g = \sqrt{81} = 9$.

Answer: (E)

14. The perimeter of quadrilateral $PQRS$ equals $PQ + QR + RS + SP$.
Since the dots are spaced 1 unit apart horizontally and vertically, then $PQ = 4$, $QR = 4$, and $PS = 1$.
Thus, the perimeter equals $4 + 4 + RS + 1$ which equals $RS + 9$.
We need to determine the length of RS.
If we draw a horizontal line from S to point T on QR, we create a right-angled triangle STR with $ST = 4$ and $TR = 3$.
By the Pythagorean Theorem, $RS^2 = ST^2 + TR^2 = 4^2 + 3^2 = 25$.
Since $RS > 0$, then $RS = \sqrt{25} = 5$.
Thus, the perimeter of quadrilateral $PQRS$ is $5 + 9 = 14$.



Answer: (C)

15. *Solution 1*

Suppose that the team has r red helmets.
Since the team has 6 more red helmets than blue helmets, then the team has $r-6$ blue helmets.
Since the ratio of the number of red helmets to the number of blue helmets is $5:3$, then $\frac{r}{r-6}=\frac{5}{3}$ and so $3r=5(r-6)$ or $3r=5r-30$.
Therefore, $2r=30$ or $r=15$.
Thus, the team has 15 red helmets, 9 blue helmets, and $15+9=24$ helmets in total.

Solution 2

Since the ratio of the number of red helmets to the number of blue helmets equals $5:3$, then we can try multiplying both parts of this ratio by small numbers to see if we can obtain an equivalent ratio where the two parts differ by 6.
Multiplying by 2, we obtain $5:3=10:6$, which doesn't have the desired property.
Multiplying by 3, we obtain $5:3=15:9$. Since $15-9=6$, then we have found the correct number of red and blue helmets.
Therefore, the team has 15 red helmets, 9 blue helmets, and $15+9=24$ helmets in total.
(If we continue to multiply this ratio by larger numbers, the difference between the two parts gets bigger, so cannot equal 6 in a different case. In other words, the answer is unique.)

Answer: (C)

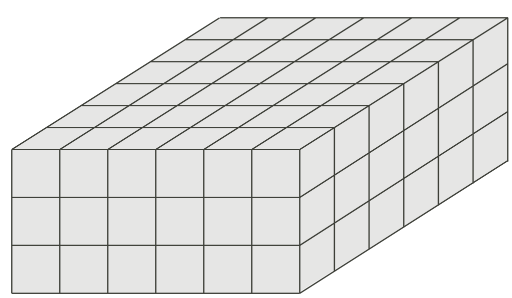
16. The quilt consists of 25 identical squares.
Of the 25 squares, 4 are entirely shaded, 8 are shaded with a single triangle that covers half of the square, and 4 are shaded with two triangles that each cover a quarter of the square.
Therefore, the shading is equivalent to the area of $4+8\times\frac{1}{2}+4\times 2\times\frac{1}{4}=10$ squares.
As a percentage, the shading is $\frac{10}{25}\times 100\%=40\%$ of the total area of the quilt.

Answer: (B)

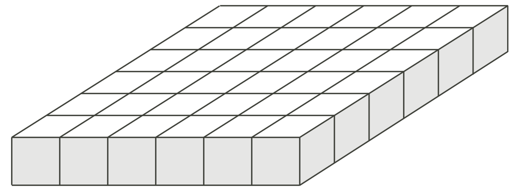
17. Since $\triangle PRS$ is isosceles with $PR=PS$, then $\angle PRS=\angle PSR$.
Since the angles in $\triangle PRS$ add to 180°, then $\angle PRS+\angle PSR+\angle RPS=180^\circ$.
Therefore, $2(\angle PRS)+34^\circ=180^\circ$ or $2(\angle PRS)=146^\circ$ or $\angle PRS=73^\circ$.
Since $\triangle PQT$ is isosceles with $PQ=PT$, then $\angle PQT=\angle PTQ=62^\circ$.
Since $\angle PRS$ is an exterior angle to $\triangle PQR$, then $\angle PRS=\angle PQR+\angle QPR$ or $73^\circ=62^\circ+x^\circ$.
Therefore, $x=73-62=11$.
(Instead, we could have determined that $\angle PRQ=180^\circ-\angle PRS=180^\circ-73^\circ=107^\circ$, and then looked at the sum of the angles in $\triangle PQR$ to get $62^\circ+x^\circ+107^\circ=180^\circ$ or $x=180-169=11$.)

Answer: (A)

18. We visualize the solid as a rectangular prism with length 6, width 6 and height 3. In other words, we can picture the solid as three 6×6 squares stacked on top of each other.
Since the entire exterior of the solid is painted, then each cube in the top layer and each cube in the bottom layer has paint on it, so we can remove these.



This leaves the middle 6×6 layer of cubes.
Each cube around the perimeter of this square has paint on it, so it is only the "middle" cubes from this layer that have no paint on them.
These middle cubes form a 4×4 square, and so there are 16 cubes with no paint on them.



Answer: (A)

19. Let $PQ = a$, $QR = b$, $PW = c$, and $WV = d$.
Since the large rectangle is divided into four smaller rectangles, then $PQ = WX = VU = a$, $QR = XS = UT = b$, $PW = QX = RS = c$, and $WV = XU = ST = d$.
Since the area of rectangle $PQXW$ is 9, then $ac = 9$.
Since the area of rectangle $QRSX$ is 10, then $bc = 10$.
Since the area of rectangle $XSTU$ is 15, then $bd = 15$.
The area of rectangle $WXUV$ is ad. We want to determine the value of ad.
If we multiply the equations $ac = 9$ and $bd = 15$, we obtain $(ac)(bd) = 9 \times 15$, or $abcd = 135$.
We divided this equation by the equation $bc = 10$.
This gives $\dfrac{abcd}{bc} = \dfrac{135}{10} = \dfrac{27}{2}$, from which $ad = \dfrac{27}{2}$.

Answer: (B)

20. *Solution 1*
When N is divided by 10, 11 or 12, the remainder is 7.
This means that $M = N - 7$ is divisible by each of 10, 11 and 12.
Since M is divisible by each of 10, 11 and 12, then M is divisible by the least common multiple of 10, 11 and 12.
Since $10 = 2 \times 5$, $12 = 2 \times 2 \times 3$, and 11 is prime, then the least common multiple of 10, 11 and 12 is $2 \times 2 \times 3 \times 5 \times 11 = 660$. (To find the least common multiple, we compute the product of the highest powers of each of the prime factors that occur in the given numbers.)
Since M is divisible by 660 and $N = M + 7$ is a three-digit positive integer, then M must equal 660. (The next largest multiple of 660 is 1320.)
Therefore, $N = M + 7 = 667$, and so the sum of the digits of N is $6 + 6 + 7 = 19$.

Solution 2
When N is divided by 10, 11 or 12, the remainder is 7.
This means that $M = N - 7$ is divisible by each of 10, 11 and 12.
Since M is divisible by each of 10 and 11, then M must be divisible by 110.
We test the first few multiples of 110 until we obtain one that is divisible by 12.
The integers 110, 220, 330, 440, and 550 are not divisible by 12, but 660 is.
Therefore, M could be 660. (This means that M must be 660.)
Finally, $N = M + 7 = 667$, and so the sum of the digits of N is $6 + 6 + 7 = 19$.

Answer: (E)

21. Let L be the length of the string.
If x is the length of the shortest piece, then since each of the other pieces is twice the length of the next smaller piece, then the lengths of the remaining pieces are $2x$, $4x$, and $8x$.
Since these four pieces make up the full length of the string, then $x + 2x + 4x + 8x = L$ or $15x = L$ and so $x = \frac{1}{15}L$.
Thus, the longest piece has length $8x = \frac{8}{15}L$, which is $\frac{8}{15}$ of the length of the string.

Answer: (A)

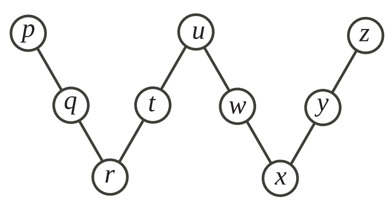
22. Suppose that the radius of each of the circles is r.
Since the two circles are identical, then the two circles have equal area.
Since the shaded area is common to the two circles, then the unshaded pieces of each circle have equal areas.
Since the combined area of the unshaded regions equals that of the shaded region, or 216π, then each of the unshaded regions have area $\frac{1}{2} \times 216\pi = 108\pi$.
The total area of one of the circles equals the sum of the areas of the shaded region and one unshaded region, or $216\pi + 108\pi = 324\pi$.
Since the radius of the circle is r, then $\pi r^2 = 324\pi$ or $r^2 = 324$.
Since $r > 0$, then $r = \sqrt{324} = 18$.
Therefore, the circumference of each circle is $2\pi r = 2\pi(18) = 36\pi$.

Answer: (C)

23. Suppose that the height of the water in each container is h cm.
Since the first container is a rectangular prism with a base that is 2 cm by 4 cm, then the volume of the water that it contains, in cm^3, is $2 \times 4 \times h = 8h$.
Since the second container is a right cylinder with a radius of 1 cm, then the volume of the water that it contains, in cm^3, is $\pi \times 1^2 \times h = \pi h$.
Since the combined volume of the water is 80 cm^3, then $8h + \pi h = 80$.
Thus, $h(8 + \pi) = 80$ or $h = \dfrac{80}{8+\pi} \approx 7.18$.
Of the given answers, this is closest to 7.2 (that is, the height of the water is closest to 7.2 cm).

Answer: (B)

24. If we have a configuration of the numbers that has the required property, then we can add or subtract the same number from each of the numbers in the circles and maintain the property. (This is because there are the same number of circles in each line.)
Therefore, we can subtract 2012 from all of the numbers and try to complete the diagram using the integers from 0 to 8.
We label the circles as shown in the diagram, and call S the sum of the three integers along any one of the lines.

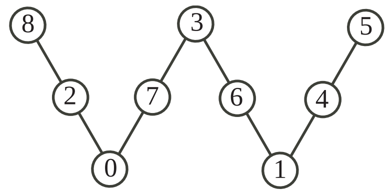


Since $p, q, r, t, u, w, x, y, z$ are 0 through 8 in some order, then

$$p + q + r + t + u + w + x + y + z = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$$

From the desired property, we want $S = p + q + r = r + t + u = u + w + x = x + y + z$.
Therefore, $(p + q + r) + (r + t + u) + (u + w + x) + (x + y + z) = 4S$.
From this, $(p+q+r+t+u+w+x+y+z)+r+u+x = 4S$ or $r+u+x = 4S-36 = 4(S-9)$.
We note that the right side is an integer that is divisible by 4.
Also, we want S to be as small as possible so we want the sum $r + u + x$ to be as small as possible.
Since $r + u + x$ is a positive integer that is divisible by 4, then the smallest that it can be is $r + u + x = 4$.

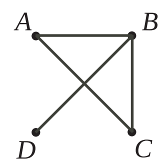
If $r+u+x=4$, then r, u and x must be 0, 1 and 3 in some order since each of r, u and x is a different integer between 0 and 8.
In this case, $4=4S-36$ and so $S=10$.
Since $S=10$, then we cannot have r and u or u and x equal to 0 and 1 in some order, or else the third number in the line would have to be 9, which is not possible.
This tells us that u must be 3, and r and x are 0 and 1 in some order.
Here is a configuration that works:



Therefore, the value of u in the original configuration is $3+2012=2015$.

Answer: (D)

25. We label the four people in the room A, B, C, and D. We represent each person by a point. There are six possible pairs of friends: AB, AC, AD, BC, BD, and CD. We represent a friendship by joining the corresponding pair of points and a non-friendship by not joining the pair of points.
Since each pair of points is either joined or not joined and there are 6 possible pairs, then there are $2\times2\times2\times2\times2\times2=2^6=64$ possible configurations of line segments in the diagram.
For example, if A and B, B and C, A and C, and B and D are friends, then we obtain the configuration to the right.
Since each pair of points has equal probability of being joined or not joined, then each of the 64 possible configurations is equally likely, so has probability $\frac{1}{64}$.

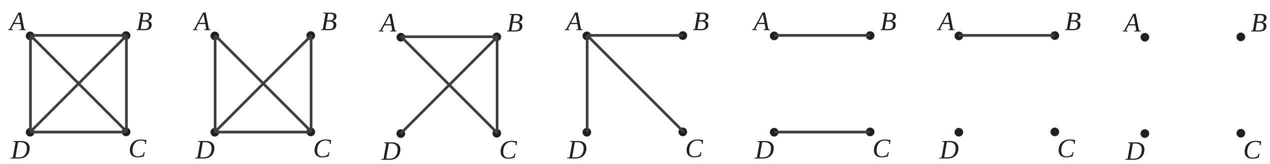


Points A and B, for example, are "connected" according to the definition, if they are joined, or if they are both joined to C, or if they are both joined to D, or if one is joined to C, the other to D and C and D are joined. In other words, points A and B are connected if we can pass from A to B along line segments, possibly through one or both of C and D.
If each pair of points is connected, we call the configuration *fully connected*.
Thus, we need to count the number of configurations that are fully connected.
First, we count the number of configurations including each possible number of line segments (0 through 6). After this, we will analyze each case to see which configurations are fully connected.

- 0 line segments: There is 1 such configuration.
- 6 line segments: There is 1 such configuration.
- 1 line segment: There are 6 such configurations, because we can choose any one of the 6 possible segments.
- 5 line segments: There are 6 such configurations, because we can start with all 6 line segments and remove any one of the 6 possible segments.
- 2 line segments: There are 15 such configurations. This is because there are 6 possible choices for the first line segment to add, and then 5 possible choices for the second line segment. This looks like 6×5 configurations, but each configuration is counted twice here (as we could choose AB and then BD or BD and then AB). Therefore, there are $\frac{1}{2}\times6\times5=15$ configurations.

- 4 line segments: There are 15 such configurations, because we can start with all 6 line segments and remove any of the 15 possible pairs of 2 line segments.
- 3 line segments: There are 20 configurations, because there are 64 configurations in total and we have already accounted for $1 + 1 + 6 + 6 + 15 + 15 = 44$ configurations.

Here is an example of each of these types of configurations:



6 segments 5 segments 4 segments 3 segments 2 segments 1 segment 0 segments

Now we analyze each of the possible classes of configurations to see if the relevant configurations are fully connected or not:

- <u>0 line segments</u>
  This configuration is not fully connected.
- <u>6 line segments</u>
  This configuration is fully connected.
- <u>1 line segment</u>
  Each of these 6 configurations is not fully connected, since it can only have 2 points joined.
- <u>5 line segments</u>
  Each of these 6 configurations is fully connected, since only one pair of points is not joined. If this pair is AB, for instance, then A and B are each joined to C, so the configuration is fully connected. (See the diagram labelled "5 segments" above.) The same is true no matter which pair is not joined.
- <u>2 line segments</u>
  Each of these 15 configurations is not fully connected, since the two line segments can only include 3 points with the fourth point not connected to any other point (for example, B connected to each of A and C) or two pairs of connected points (as in the diagram labelled "2 segments" above).
- <u>4 line segments</u>
  Each of these 15 configurations is fully connected.
  Consider starting with all 6 pairs of points joined, and then remove two line segments.
  There are two possibilities: either the two line segments share an endpoint, or they do not.
  An example of the first possibility is removing AB and BC to get Figure 1. This configuration is fully connected, and is representative of this subcase.
  An example of the second possibility is removing AB and CD to get Figure 2. This configuration is fully connected, and is representative of this subcase.

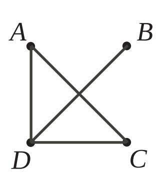


Figure 1

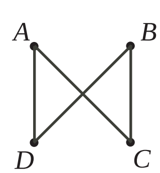


Figure 2

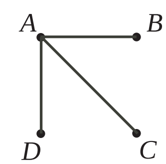
Therefore, all 15 of these configurations are fully connected.

– 3 line segments

There are 20 such configurations.
Each of these configurations involve joining more than 2 points, since 2 points can only be joined by 1 line segment.
There are several possibilities:

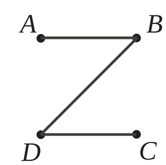
∗ Some of these 20 configurations involve only 3 points. Since there are three line segments that join 3 points, these configurations look like a triangle and these are not fully connected. There are 4 such configurations, which we can see by choosing 1 of 4 points to not include.

∗ Some of the remaining 16 configurations involve connecting 1 point to each of the other 3 points. Without loss of generality, suppose that A is connected to each of the other points.
This configuration is fully connected since B and C are connected through A, as are B and D, C and D, and is representative of this subcase.

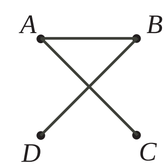


There are 4 such configurations, which we can see by choosing 1 of 4 points to be connected to each of the other points.

∗ Consider the remaining $20 - 4 - 4 = 12$ configurations. Each of these involves all 4 points and cannot have 1 point connected to the other 3 points.
Without loss of generality, we start by joining AB.
If CD is joined, then one of AC, AD, BC, and BD is joined, which means that each of A and B is connected to each of C and D.
This type of configuration is fully connected.



If CD is not joined, then two of AC, AD, BC, and BD are joined.
We cannot have AC and AD both joined, or BC and BD both joined, since we cannot have 3 segments coming from the same point.
Also, we cannot have AC and BC both joined, or AD and BD both joined, since we cannot include just 3 points.
Therefore, we must have AC and BD joined, or AD and BC joined.
This type of configuration is fully connected.



Therefore, of the 64 possible configurations, $1 + 6 + 15 + 4 + 12 = 38$ are fully connected.
Therefore, the probability that every pair of people in this room is connected is $\frac{38}{64} = \frac{19}{32}$.

ANSWER: (D)