2012 Hypatia Contest

Thursday, April 12, 2012
(in North America and South America)

Friday, April 13, 2012
(outside of North America and South America)

Solutions
1. (a) In \( \triangle PTQ \), \( \angle PTQ = 90^\circ \).
Using the Pythagorean Theorem, \( PQ^2 = 32^2 + 24^2 \) or \( PQ^2 = 1024 + 576 = 1600 \) and so \( PQ = \sqrt{1600} = 40 \), since \( PQ > 0 \).
(b) In \( \triangle QTR \), \( \angle QTR = 90^\circ \).
Using the Pythagorean Theorem, \( 51^2 = TR^2 + 24^2 \) or \( TR^2 = 2601 - 576 = 2025 \) and so \( TR = \sqrt{2025} = 45 \), since \( TR > 0 \).
Since \( PT = 32 \) and \( TR = 45 \), then \( PR = PT + TR = 32 + 45 = 77 \).
In \( \triangle PQR \), \( QT \) is perpendicular to base \( PR \) and so \( \triangle PQR \) has area \( \frac{1}{2} \times (PR) \times (QT) = \frac{1}{2} \times 77 \times 24 = 924 \).
(c) From part (b), the length of \( PR \) is 77.
Since \( QS : PR = 12 : 11 \), then \( QS = 77 \times \frac{12}{11} = 84 \).
So then \( TS = QS - QT = 84 - 24 = 60 \).
Using the Pythagorean Theorem in \( \triangle PTS \), \( PS = \sqrt{32^2 + 60^2} \) or \( PS = \sqrt{4624} = 68 \), since \( PS > 0 \).
Using the Pythagorean Theorem in \( \triangle RTS \), \( RS = \sqrt{45^2 + 60^2} \) or \( RS = \sqrt{5625} = 75 \), since \( RS > 0 \).
Thus, quadrilateral \( PQRS \) has perimeter \( 40 + 51 + 75 + 68 = 234 \).

2. (a) Expanding, \( (a + b)^2 = a^2 + 2ab + b^2 = (a^2 + b^2) + 2ab \).
Since \( a^2 + b^2 = 24 \) and \( ab = 6 \), then \( (a + b)^2 = 24 + 2(6) = 36 \).
(b) Expanding, \( (x + y)^2 = x^2 + 2xy + y^2 = (x^2 + y^2) + 2xy \).
Since \( x^2 + y^2 = 13 \) and \( x^2 + y^2 = 7 \), then \( 13 = 7 + 2xy \) or \( 2xy = 6 \), and so \( xy = 3 \).
(c) Expanding, \( (j + k)^2 = j^2 + 2jk + k^2 = (j^2 + k^2) + 2jk \).
Since \( j + k = 6 \) and \( j^2 + k^2 = 52 \), then \( 6^2 = 52 + 2jk \) or \( 2jk = -16 \), and so \( jk = -8 \).
(d) Expanding, \( (m^2 + n^2)^2 = m^4 + 2m^2n^2 + n^4 = (m^4 + n^4) + 2m^2n^2 \).
Since \( m^2 + n^2 = 12 \) and \( m^4 + n^4 = 136 \), then \( 12^2 = 136+2m^2n^2 \) or \( 2m^2n^2 = 8 \) or \( m^2n^2 = 4 \), and so \( mn = \pm 2 \).

3. (a) Since \( \angle MON = 90^\circ \), the product of the slopes of \( NO \) and \( OM \) is \(-1\).
The slope of \( NO \) is \( \frac{n^2 - 0}{n - 0} = n \), since \( n \neq 0 \) (points \( N \) and \( O \) are distinct).
The slope of \( OM \) is \( \frac{\frac{1}{2} - 0}{\frac{1}{2} - 0} = \frac{1}{2} \).
Thus, \( n \times \frac{1}{2} = -1 \) or \( n = -2 \).
(b) Since \( \angle ABO = 90^\circ \), the product of the slopes of \( BA \) and \( BO \) is \(-1\).
The slope of \( BA \) is \( \frac{b^2 - 4}{b - 2} = \frac{(b - 2)(b + 2)}{b - 2} = b + 2 \), since \( b \neq 2 \) (\( A \) and \( B \) are distinct).
The slope of \( BO \) is \( \frac{b^2 - 0}{b - 0} = b \), since \( b \neq 0 \) (\( B \) and \( O \) are distinct).
Thus, \( (b + 2) \times b = -1 \) or \( b^2 + 2b + 1 = 0 \).
Factoring, \( (b + 1)(b + 1) = 0 \) and so \( b = -1 \).
(c) Since \( \angle PQR = 90^\circ \), the product of the slopes of \( PQ \) and \( RQ \) is \(-1\).
The slope of \( PQ \) is \( \frac{p^2 - q^2}{p - q} = \frac{(p - q)(p + q)}{p - q} = p + q \), since \( p \neq q \) (\( P \) and \( Q \) are distinct).
The slope of \( RQ \) is \( \frac{r^2 - q^2}{r - q} = \frac{(r - q)(r + q)}{r - q} = r + q \), since \( r \neq q \) (\( R \) and \( Q \) are distinct).
Thus, \((p + q) \times (r + q) = -1\).
Since \(p, q\) and \(r\) are integers, then \(p + q\) and \(r + q\) are integers.
In order that \((p + q) \times (r + q) = -1\), either \(p + q = 1\) and \(r + q = -1\) or \(p + q = -1\) and \(r + q = 1\) (these are the only possibilities for integers \(p, q, r\) for which \((p + q) \times (r + q) = -1\).
In the first case, we add the two equations to get \(p + q + r + q = 1 + (-1)\) or \(2q + p + r = 0\).
In the second case, adding the two equations gives \(p + q + r + q = -1 + 1\) or \(2q + p + r = 0\).
In either case, \(2q + p + r = 0\), as required.

4. (a) Since \(p\) is an odd prime integer, then \(p > 2\).
Since the only prime divisors of \(2p^2\) are 2 and \(p\), then the positive divisors of \(2p^2\) are \(1, 2, p, 2p, p^2,\) and \(2p^2\).
So then, \(S(2p^2) = 1 + 2 + p + 2p + p^2 + 2p^2 = 3p^2 + 3p + 3\).
Since \(S(2p^2) = 2613\), then \(3p^2 + 3p + 3 = 2613\) or \(3p^2 + 3p - 2610 = 0\) or \(p^2 + p - 870 = 0\).
Factoring, \((p + 30)(p - 29) = 0\), and so \(p = 29\) (\(p \neq -30\) since \(p\) is an odd prime).

(b) Suppose \(m = 2p\) and \(n = 9q\) for some prime numbers \(p, q > 3\).
The positive divisors of \(2p\), thus \(m\), are \(1, 2, p, 2p\) (since \(p > 3\)).
Therefore, \(S(m) = 1 + 2 + p + 2p = 3p + 3\).
The positive divisors of \(9q\), thus \(n\), are \(1, 3, q, 3q, 9,\) and \(9q\) (since \(q > 3\)).
Therefore, \(S(n) = 1 + 3 + 9 + q + 3q + 9q = 13q + 13\).
Since \(S(m) = S(n)\), then \(3p + 3 = 13q + 13\) or \(3p - 13q = 10\).
Also, \(m\) and \(n\) are consecutive integers and so either \(m - n = 1\) or \(n - m = 1\).

If \(m - n = 1\), then \(2p - 9q = 1\).
We solve the following system of two equations and two unknowns.

\[
\begin{align*}
2p - 9q &= 1 \\
3p - 13q &= 10
\end{align*}
\]

Multiplying equation (1) by 3 and equation (2) by 2 we get,

\[
\begin{align*}
6p - 27q &= 3 \\
6p - 26q &= 20
\end{align*}
\]

Subtracting equation (3) from equation (4), we get \(q = 17\).
Substituting \(q = 17\) into equation (1), \(2p - 9(17) = 1\) or \(2p = 154\), and so \(p = 77\).
However, \(p\) must be a prime and thus \(p \neq 77\).
There is no solution when \(m - n = 1\).

If \(n - m = 1\), then \(9q - 2p = 1\).
We solve the following system of two equations and two unknowns.

\[
\begin{align*}
9q - 2p &= 1 \\
3p - 13q &= 10
\end{align*}
\]

Multiplying equation (5) by 3 and equation (6) by 2 we get,

\[
\begin{align*}
27q - 6p &= 3 \\
6p - 26q &= 20
\end{align*}
\]

Adding equation (7) and equation (8), we get \(q = 23\).
Substituting \(q = 23\) into equation (6), \(3p - 13(23) = 10\) or \(3p = 309\), and so \(p = 103\).
Since \(q = 23\) and \(p = 103\) are prime integers greater than 3, then \(m = 2(103) = 206\) and \(n = 9(23) = 207\) are the only pair of consecutive integers satisfying the given properties.
(c) Since the only prime divisors of $p^3q$ are $p$ and $q$, then the positive divisors of $p^3q$, are $1, p, q, pq, p^2, q^2, p^3, q^3$, and $p^3q$ (since $p$ and $q$ are distinct primes).

Therefore, $S(p^3q) = p^3q + p^3 + p^2q + p^2 + pq + p + q + 1$.

Simplifying,

\[
S(p^3q) = p^3q + p^3 + p^2q + p^2 + pq + p + q + 1
= (p^3q + p^2q + pq + q) + (p^3 + p^2 + p + 1)
= q(p^3 + p^2 + p + 1) + (p^3 + p^2 + p + 1)
= (q + 1)(p^3 + p^2 + p + 1)
= (q + 1)(p^2(p + 1) + (p + 1))
= (q + 1)(p + 1)(p^2 + 1)
\]

We are to determine the number of pairs of distinct primes $p$ and $q$, each less than 30, such that $(q + 1)(p + 1)(p^2 + 1)$ is not divisible by 24.

There are 10 primes less than 30. These are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29.

Therefore, the total number of possible pairs $(p, q)$, where $p \neq q$, is $10 \times 9 = 90$.

We will count the number of pairs $(p, q)$ for which $(q + 1)(p + 1)(p^2 + 1)$ is divisible by 24 and then subtract this total from 90.

If $p$ or $q$ equals 23, then 24 divides $(q + 1)(p + 1)(p^2 + 1)$.

There are 9 ordered pairs of the form $(23, q)$ and 9 of the form $(p, 23)$.

Thus, we count 18 pairs and since we have exhausted all possibilities using 23, we remove it from our list of 10 primes above.

Since $24 = 2^3 \times 3$, we can determine values of $q$ for a given value of $p$ by recognizing that each of these prime factors (three 2s and one 3) must occur in the prime factorization of $(q + 1)(p + 1)(p^2 + 1)$.

For example if $p = 2$, then $(q + 1)(p + 1)(p^2 + 1) = (q + 1)(3)(5)$.

Therefore, for $(q + 1)(p + 1)(p^2 + 1)$ to be a multiple of 24, $q + 1$ must be a multiple of 8 (since we are missing $2^3$).

Thus when $p = 2$, the only possible value of $q$ is 7 (we get this by trying the other 8 values in the list of primes).

We organize all possibilities for $p$ (and the resulting values of $q$) in the table below.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$(p + 1)(p^2 + 1)$</th>
<th>$q + 1$ must be a multiple of</th>
<th>$q$ (distinct from $p$)</th>
<th>Number of ordered pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$(3)(5)$</td>
<td>$2^3 = 8$</td>
<td>$q = 7$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$(4)(10) = 2^4 \times 5$</td>
<td>3</td>
<td>$q = 2, 5, 11, 17, 29$</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>$(6)(26) = 2^2 \times 3 \times 13$</td>
<td>2</td>
<td>$q = 3, 7, 11, 13, 17, 19, 29$</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>$(8)(50) = 2^4 \times 50$</td>
<td>3</td>
<td>$q = 2, 5, 11, 17, 29$</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>$(12)(122) = 2^4 \times 3 \times 61$</td>
<td>any $q$ will work</td>
<td>$q = 2, 3, 5, 7, 13, 17, 19, 29$</td>
<td>8</td>
</tr>
<tr>
<td>13</td>
<td>$(14)(170) = 2^2 \times 595$</td>
<td>$2 \times 3 = 6$</td>
<td>$q = 5, 11, 17, 29$</td>
<td>4</td>
</tr>
<tr>
<td>17</td>
<td>$(18)(290) = 2^2 \times 3 \times 435$</td>
<td>2</td>
<td>$q = 3, 5, 7, 11, 13, 19, 29$</td>
<td>7</td>
</tr>
<tr>
<td>19</td>
<td>$(20)(362) = 2^3 \times 905$</td>
<td>3</td>
<td>$q = 2, 5, 11, 17, 29$</td>
<td>5</td>
</tr>
<tr>
<td>29</td>
<td>$(30)(842) = 2^2 \times 3 \times 2105$</td>
<td>2</td>
<td>$q = 3, 5, 7, 11, 13, 17, 19$</td>
<td>7</td>
</tr>
</tbody>
</table>

The total number of pairs $(p, q)$ for which 24 divides $S(p^3q)$ is

$$18 + 1 + 5 + 7 + 5 + 8 + 4 + 7 + 5 + 7 = 67.$$  

Thus, the total number of pairs of distinct prime integers $p$ and $q$, each less than 30, such that $S(p^3q)$ is not divisible by 24, is $90 - 67 = 23$. 
