The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2012 Galois Contest

Thursday, April 12, 2012
(in North America and South America)

Friday, April 13, 2012
(outside of North America and South America)

Solutions

1. (a) In $\triangle JPA$, $\angle JPA = 90^\circ$.
Using the Pythagorean Theorem, $AJ^2 = 15^2 + 20^2$ or $AJ^2 = 225 + 400 = 625$ and so $AJ = \sqrt{625} = 25$, since $AJ > 0$.
The distance from A to J is 25.

(b) In $\triangle BAQ$, $\angle BAQ = 90^\circ$.
Using the Pythagorean Theorem, $39^2 = BA^2 + 15^2$ or $BA^2 = 1521 - 225 = 1296$ and so $BA = \sqrt{1296} = 36$, since $BA > 0$.
The distance from B to A is 36.

(c) In $\triangle BJA$, $\angle BJA = 90^\circ$.
Using the Pythagorean Theorem, $BA^2 = BJ^2 + AJ^2$ or $1296 = BJ^2 + 625$ or $BJ^2 = 1296 - 625 = 671$ and so $BJ = \sqrt{671} \approx 25.904$, since $BJ > 0$.
Thus, the distance from Budan's ball to the jack ball is approximately 25.904.
In part (a), we determined the distance between Adam's ball and the jack ball to be 25.
Therefore, Adam's ball is closer to the jack ball.

2. (a) The average of any two numbers is found by adding the two numbers and dividing by two.
Thus, the three averages when the numbers are taken in pairs are

$$\frac{25+5}{2} = \frac{30}{2} = 15,\ \ \frac{5+29}{2} = \frac{34}{2} = 17,\ \text{and}\ \frac{25+29}{2} = \frac{54}{2} = 27.$$

(b) The average of 2 and 6 is $\frac{2+6}{2} = 4$.
Since 6 is greater than 2, then the average of 6 and n is greater than the average of 2 and n.
Therefore, the average of 6 and n is 13 and the average of 2 and n is 11.
Since the average of 6 and n is 13, then $\frac{6+n}{2} = 13$ or $6 + n = 26$ and so $n = 20$.
We can check that $n = 20$ is correct by recognizing that the average of 2 and 20 is indeed 11.

(c) When each of the three numbers is added to the average of the other two, the resulting three expressions are

$$2 + \frac{a+b}{2},\ \ a + \frac{2+b}{2},\ \ b + \frac{2+a}{2}.$$

To determine which of these expressions is equal to which of the results, 14, 17, 21, we must order the three expressions from smallest to largest.
Since $2 < a < b$, then $2 + (2 + a + b) < a + (2 + a + b) < b + (2 + a + b)$
or $4 + a + b < 2a + 2 + b < 2b + 2 + a$.
Dividing by 2, $\frac{4+a+b}{2} < \frac{2a+2+b}{2} < \frac{2b+2+a}{2}$ or $\frac{4}{2}+\frac{a+b}{2} < \frac{2a}{2}+\frac{2+b}{2} < \frac{2b}{2}+\frac{2+a}{2}$
and so $2 + \frac{a+b}{2} < a + \frac{2+b}{2} < b + \frac{2+a}{2}$.
Since $2 + \frac{a+b}{2}$ is the smallest of the three expressions, then it must equal the smallest of the three results, 14.
Since $b + \frac{2+a}{2}$ is the largest of the three expressions, then it must equal the largest of the three results, 21.

We now solve the following system of two equations and two unknowns.

$$2 + \frac{a+b}{2} = 14 \qquad (1)$$

$$b + \frac{2+a}{2} = 21 \qquad (2)$$

Multiplying each equation by 2,

$$4 + a + b = 28 \qquad (3)$$

$$2b + 2 + a = 42 \qquad (4)$$

Thus,

$$a + b = 24 \qquad (5)$$

$$a + 2b = 40 \qquad (6)$$

Subtracting equation (5) from equation (6), we get $b = 16$.
Substituting $b = 16$ into equation (5), $a + 16 = 24$, and so $a = 8$.
(We may check that our solution is correct by substituting $a = 8$ and $b = 16$ into the third expression $a + \frac{2+b}{2}$ to get the third result, 17.)

3. (a) The slope of the line is $m = -3$; thus its equation is $y = -3x + b$ with y-intercept b.
Since the line passes through the point $(2, 6)$, then $x = 2$ and $y = 6$ satisfy the equation of the line.
Substituting $x = 2$ and $y = 6$ into the equation of the line, then $6 = -3(2) + b$ and so $b = 12$.
The equation of the line is $y = -3x + 12$ and the line has y-intercept 12.
To find the x-intercept, we let $y = 0$ and solve for x.
Thus, $0 = -3x + 12$ or $3x = 12$, and so the line has x-intercept 4.

(b) The slope of the line is m; thus its equation is $y = mx + b$ with y-intercept b.
Since the line passes through the point $(2, 6)$, then $x = 2$ and $y = 6$ satisfy the equation of the line.
Substituting $x = 2$ and $y = 6$ into the equation of the line, then $6 = 2m + b$ and so $b = 6 - 2m$.
The equation of the line is $y = mx + (6 - 2m)$ and the line has y-intercept $6 - 2m$.
To find the x-intercept, we let $y = 0$ and solve for x.
Thus, $0 = mx + (6 - 2m)$ or $mx = 2m - 6$ or $x = \frac{2m-6}{m}$, and so the line has x-intercept $2 - \frac{6}{m}$.
(We require $m \neq 0$, otherwise the line is horizontal and the x-intercept does not exist.)

(c) The line through the point $(2, 6)$ with slope m has x-intercept $2 - \frac{6}{m}$ and y-intercept $6 - 2m$, as determined in part (b).
(We require $m \neq 0$, otherwise the line is horizontal and the x-intercept, P, does not exist.)
Since P is the x-intercept of this line, OP has length $2 - \frac{6}{m}$.
Since Q is the y-intercept of this line, OQ has length $6 - 2m$.
Therefore, the area of $\triangle POQ$ is given by $\frac{1}{2}(OP)(OQ) = \frac{1}{2}\left(2 - \frac{6}{m}\right)(6 - 2m)$.

Since the area of $\triangle POQ$ is 25, then $\frac{1}{2}\left(2-\frac{6}{m}\right)(6-2m)=25$.
Solving for m,

$$\begin{aligned}
\frac{1}{2}\left(2-\frac{6}{m}\right)(6-2m) &= 25 \\
\left(2-\frac{6}{m}\right)(6-2m) &= 50 \\
(2m-6)(6-2m) &= 50m \\
12m-4m^2-36+12m &= 50m \\
4m^2+26m+36 &= 0 \\
2m^2+13m+18 &= 0 \\
(2m+9)(m+2) &= 0
\end{aligned}$$

Therefore, two possible values are $m=-\frac{9}{2}$ and $m=-2$.
Since P and Q lie on the positive x-axis and the positive y-axis respectively, we must check that these two values for m give $2-\frac{6}{m}>0$ and $6-2m>0$.
When $m=-\frac{9}{2}$, $2-\frac{6}{m}=2+\frac{6}{\frac{9}{2}}$ which is greater than 0.
When $m=-\frac{9}{2}$, $6-2m=6+2(\frac{9}{2})$ which is also greater than 0.
When $m=-2$, $2-\frac{6}{m}=2+\frac{6}{2}$ which is greater than 0.
When $m=-2$, $6-2m=6+2(2)$ which is also greater than 0.
Therefore, the two values of m for which P and Q lie on the positive x-axis and the positive y-axis, respectively, and for which $\triangle POQ$ has area 25, are $m=-\frac{9}{2}$ and $m=-2$.
Note: If we remove the restriction that P and Q both be located on their respective *positive* axes, then there are two more values of m for which $\triangle POQ$ has area 25. Can you determine these?

4. (a) Let point P be the location where the students should meet such that the total distance travelled by all five is as small as possible.
Label the intersections where Abe, Bo, Carla, Denise, and Ernie initially begin, A, B, C, D, and E, respectively.

If P is located west (left) of A, then every student must walk farther to reach P than they would if P was moved to A.
Similarly, if P is located east (right) of E, then every student must walk farther to reach P than they would if P was moved to E.
That is, P must lie on the east-west street somewhere from A to E, inclusive.
To minimize the total distance travelled by all five students, we need to minimize $AP+BP+CP+DP+EP$.
For any point P located between A and E inclusive, $AP+EP=14$, since A and E combined walk the entire length of the road to meet at P.
Thus, we need to minimize $BP+CP+DP$.

If P is located west (left) of B, then the three students Bo, Carla and Denise must walk farther to reach P than they would if P was moved to B (and the combined distance for A and E is still 14).
Similarly, if P is located east (right) of D, then these three students must walk farther to reach P than they would if P was moved to D.
That is, P must lie on the east-west street somewhere from B to D, inclusive.
For any such point P located between B and D inclusive, $BP + DP = 6$.
Thus, we need to minimize CP.
This is done by locating P at C such that $CP = 0$.

The minimum total distance travelled by all five students is $14 + 6$ or 20, and this only occurs when the students meet at C, the intersection at which Carla begins.

(b) Let the number of students be $2n$ (since there are an even number of them).
Let point P be the location where the students should meet such that the total distance travelled by all is as small as possible.
Label the students and their starting intersections in order from 1 to $2n$ beginning with the student who is farthest north, as shown.

As in part (a), if P is located north of Student 1, then every student must walk farther to reach P than they would if P was moved to Intersection 1.
Similarly, if P is located south of Student $2n$, then every student must walk farther to reach P than they would if P was moved to Intersection $2n$.
Thus, we may conclude that P must lie on the north-south street somewhere from Intersection 1 to Intersection $2n$, inclusive.
For any such location of P, the combined distance travelled by Student 1 and Student $2n$ is constant (it's the distance between Intersection 1 and Intersection $2n$).
Thus, to minimize the total distance travelled by all students, we must minimize the distance travelled by Students $2, 3, 4, \ldots, (2n - 1)$.

Again, if P is located north of Student 2, then each of these $(2n - 2)$ students must walk farther to reach P than they would if P was moved to Intersection 2.
Similarly, if P is located south of Student $(2n - 1)$, then each of these students must walk farther to reach P than they would if P was moved to Intersection $(2n - 1)$.
Thus, we may conclude that P must lie on the north-south street somewhere from Intersection 2 to Intersection $(2n - 1)$, inclusive.
For any such location of P, the combined distance travelled by Student 2 and Student $(2n - 1)$ is constant (it's the distance between Intersection 2 and Intersection $(2n - 1)$).
Thus, to minimize the total distance travelled by all students, we must minimize the distance travelled by Students $3, 4, 5, \ldots, (2n - 2)$.

We may continue the argument in this manner, explaining why P must next be located between Intersection 3 and Intersection $(2n - 2)$, then Intersection 4 and Intersection $(2n - 3)$, then Intersection 5 and Intersection $(2n - 4)$, and so on.
Ultimately we conclude that P must be located between the middle two intersections, Intersection n and Intersection $(n + 1)$, inclusive (since there are $(n - 1)$ intersections north of Intersection n and $(n - 1)$ intersections south of Intersection $(n + 1)$).
For any such location of P, the combined distance travelled by Student n and Student $(n + 1)$ is constant (it's the distance between Intersection n and Intersection $(n + 1)$).
Thus, to minimize the total distance travelled by all students, they should meet anywhere between Intersection n and Intersection $(n + 1)$, inclusive.

(c) Since each student must walk along the streets, their total distance travelled is the sum of their distance travelled east or west and their distance travelled north or south.
Further, if for example a student is required to travel 5 km east and 4 km north, this may be achieved by travelling along many different paths.
However, the one thing that all these routes of minimum distance have in common is that the total distance travelled east is 5 km and the total distance travelled north is 4 km.
That is, the distance travelled east-west is independent of the distance travelled north-south, and as such, we may minimize them independently in order to arrive at the minimum total distance travelled.
Strictly speaking, this works only if we can minimize both distances at the same point, which we can here.
This reduces the problem into two distinct parts.
In Part 1, we will find the east-west location (an x-coordinate) that minimizes the total distance that the students must travel horizontally in the plane.
In Part 2, we will find the north-south location (a y-coordinate) that minimizes the total distance that the students must travel vertically in the plane.
We can then combine these to determine one location that minimizes both, so minimizes the total distance.

Part 1
Since we are attempting to locate the intersection that minimizes total east-west travel for all 100 students, we need only consider their starting east-west locations, that is, their x-coordinates.
The x-coordinates for the first 50 students are, $2, 4, 8, 16, 32, 64, \ldots, 2^{49}, 2^{50}$.
The x-coordinates for the students numbered 51 to 100 are, $1, 2, 3, 4, 5, 6, \ldots, 49, 50$.
Finding the x location that will make the total horizontal distance travelled as small as possible, is equivalent to the problem that we solved in part (b).
We must order the 100 x-coordinates from lowest to highest and then determine the two x values that are in the middle of the ordered list (as we did in part (b)).
Since there are 100 x values, we are looking for the 50th and 51st numbers in the ordered list.
There are 5 numbers less than 45 in the list $2, 4, 8, 16, 32, 64, \ldots, 2^{49}, 2^{50}$.
These 5 numbers, $2, 4, 8, 16, 32$ along with the 44 numbers $1, 2, 3, 4, 5, 6, \ldots, 44$ from the second list, will be the first 49 numbers in the ordered list.
Thus, the 50th and 51st numbers are 45 and 46.
These x-coordinates are the horizontal positions that will minimize the total distance travelled by all students in an east-west direction.

Part 2
Since we are attempting to locate the intersection that minimizes total north-south travel for all 100 students, we need only consider their starting north-south locations, that is, their y-coordinates.
The y-coordinates for the first 50 students are, $1, 2, 3, 4, 5, 6, \ldots, 49, 50$.
The y-coordinates for the students numbered 51 to 100 are, $2, 4, 6, 8, 10, 12, \ldots, 98, 100$.
Finding the y location that will make the total vertical distance travelled as small as possible, is also equivalent to the problem that we solved in part (b).
We must order the 100 y-coordinates from lowest to highest and then determine the two y values that are in the middle of the ordered list.
Since there are 100 y values, we are looking for the 50th and 51st numbers in the ordered

list.
There are 49 numbers in the ordered list that are less than or equal to 33 (33 of these from the first list and 16 from the second).
Thus, the 50th and 51st numbers in the ordered list are both 34 (since 34 appears in both the first list and the second).
Since the 50th and 51st numbers are both 34, there is just one y-coordinate that will minimize the total distance travelled by all students in a north-south direction.

The total horizontal travel is minimized when $x = 45$ or $x = 46$, and the total vertical travel is minimized when $y = 34$.
Thus, the intersections at which the students should meet in order to make the total distance travelled by all students as small as possible are $(45, 34)$ or $(46, 34)$.