The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2012 Fryer Contest

Thursday, April 12, 2012
(in North America and South America)

Friday, April 13, 2012
(outside of North America and South America)

Solutions

1. (a) Candidate A received $\frac{1008}{5600} \times 100\% = 0.18 \times 100\%$ or 18% of all votes.

   (b) *Solution 1*
   Since $\frac{3}{5} \times 100\% = 0.60 \times 100\%$, Candidate B received 60% of all votes.
   Since Candidates C and D tied, they equally shared the remaining $100\% - 60\% = 40\%$ of the votes.
   Therefore, Candidate C received $\frac{1}{2}$ of 40% of the votes, or 20% of all votes.

   Solution 2
   Since Candidate B received $\frac{3}{5}$ of all votes, then Candidates C and D shared the remaining $1 - \frac{3}{5} = \frac{2}{5}$ of all votes.
   Candidates C and D tied, thus they shared equally the remaining $\frac{2}{5}$ of the votes.
   Therefore, Candidate C received $\frac{1}{2}$ of $\frac{2}{5}$ or $\frac{1}{2} \times \frac{2}{5} = \frac{2}{10} = \frac{1}{5}$ of the votes.
   Since $\frac{1}{5} \times 100\% = 0.20 \times 100\%$, Candidate C received 20% of all votes.

   (c) *Solution 1*
   At 10:00 p.m., 90% of 6000 votes or $\frac{90}{100} \times 6000 = 5400$ votes had been counted.
   Of those 5400 votes that had been counted, Candidate E received 53%.
   Therefore at 10 p.m., $\frac{53}{100} \times 5400 = 2862$ votes had been counted for Candidate E.
   Since there were only 2 candidates, the remaining $5400 - 2862$ or 2538 votes must have been counted for Candidate F.
   Thus, there were $2862 - 2538$ or 324 more votes counted for Candidate E than for Candidate F.

   Solution 2
   At 10:00 p.m., 90% of 6000 votes or $\frac{90}{100} \times 6000 = 5400$ votes had been counted.
   Of those 5400 votes that had been counted, Candidate E received 53%.
   Since there are only 2 candidates, then Candidate F must have received the remaining $100\% - 53\%$ or 47%.
   Thus, Candidate E received $53\% - 47\%$ or 6% more votes than Candidate F.
   Since there were a total of 5400 votes that had been counted at 10:00p.m., then Candidate E received 6% of 5400 or 324 more votes than Candidate F.

   (d) Candidate H received 40% of the votes and Candidate J received 35% of the votes.
   Thus, the only other candidate, G, received the remaining $100\% - 40\% - 35\% = 25\%$ of the votes.
   Since Candidate G received 2000 votes representing 25% of all votes cast, then the total number of votes cast was $2000 \times 4 = 8000$ (since $25\% \times 4 = 100\%$).
   Thus, Candidate H received 40% of 8000 votes or $\frac{40}{100} \times 8000 = 3200$ votes.

2. (a) Factoring gives $112 = 2 \times 56 = 2 \times 2 \times 28 = 2 \times 2 \times 2 \times 14 = 2 \times 2 \times 2 \times 2 \times 7$.
   So, the prime factorization of 112 is $2 \times 2 \times 2 \times 2 \times 7$ or $2^4 \times 7$.

   (b) For every perfect square, each of its prime factors occurs an even number of times (see the *Note* at the end of part (d) for a brief explanation of this).
   From part (a), the prime factorization of 112 is $2^4 \times 7$.
   We are asked for the smallest value of the positive integer u so that $112 \times u$ or $2^4 \times 7 \times u$ is a perfect square.
   The prime factor 2 already occurs an even number of times, (four times), in the factorization of 112.
   Thus, no additional factors of 2 are needed to make $2^4 \times 7 \times u$ a perfect square.
   However, the prime factor 7 occurs only once.
   Since all prime factors must occur an even number of times, then at least one additional

factor of 7 is needed for $2^4 \times 7 \times u$ to be a perfect square.
Therefore, the smallest positive integer u that makes the product $112 \times u$ a perfect square, is 7:

$$112 \times u = 2^4 \times 7 \times u = 2^4 \times 7 \times 7 = 2^4 \times 7^2 = (2^2 \times 7) \times (2^2 \times 7).$$

(c) Since $5632 = 512 \times 11$ and $512 = 2^9$, then the prime factorization of 5632 is $2^9 \times 11$.
Again, every perfect square has each of its prime factors occuring an even number of times.
We are asked for the smallest value of the positive integer v so that $5632 \times v$ or $2^9 \times 11 \times v$ is a perfect square.
The prime factor 2 occurs an odd number of times, (nine times), in the factorization of 5632.
Thus, at least one additional factor of 2 is needed to make $2^9 \times 11 \times v$ a perfect square.
The prime factor 11 occurs only once.
Thus, at least one additional factor of 11 is needed for $2^9 \times 11 \times v$ to be a perfect square.
Therefore, the smallest positive integer v that makes the product $5632 \times v$ a perfect square, is 2×11 or 22:

$$5632 \times v = 2^9 \times 11 \times v = 2^9 \times 11 \times 2 \times 11 = 2^{10} \times 11^2 = (2^5 \times 11) \times (2^5 \times 11).$$

(d) For every perfect cube, the number of times that each of its prime factors occurs is a multiple of 3 (see the *Note* below for a brief explanation of this).
From part (a), the prime factorization of 112 is $2^4 \times 7$.
We are asked for the smallest value of the positive integer w so that $112 \times w$ or $2^4 \times 7 \times w$ is a perfect cube.
The prime factor 2 occurs four times in the factorization of 112.
Thus, the smallest number of additional factors of 2 needed to make $2^4 \times 7 \times w$ a perfect cube is two (since 6 is the smallest multiple of 3 that is greater than 4).
The prime factor 7 occurs only once.
Thus, the smallest number of additional factors of 7 needed to make $2^4 \times 7 \times w$ a perfect cube is two (since 3 is the smallest multiple of 3 that is greater than 1).

Therefore, the smallest positive integer w that makes the product $112 \times w$ a perfect cube, is $2^2 \times 7^2$ or 196:

$$112 \times w = 2^4 \times 7 \times w = 2^4 \times 7 \times 2^2 \times 7^2 = 2^6 \times 7^3 = (2^2 \times 7) \times (2^2 \times 7) \times (2^2 \times 7).$$

Note: Every positive integer greater than 1 can be written as a unique product of prime numbers (this is known as the Fundamental Theorem of Arithmetic!).
Every perfect square, P, is the product of a positive integer, n, with itself.
That is, $P = n \times n$.
By the Fundamental Theorem of Arithmetic, n can be written as a product of prime numbers.
Since $P = n \times n$, the prime factors of P are matching pairs of prime factors of n.
Thus, the prime factors of every perfect square occur an even number of times.
This argument similarly extends to every perfect cube, C.
Since $C = n \times n \times n$ for some positive integer n, then the prime factors of C occur in sets of three matching prime factors of n.
Thus for every perfect cube, the number of times that each of the prime factors occurs is a multiple of 3.

3. (a) The first row contains the integers 1 through 6.
Each successive row contains the next six integers, in order, that follow the largest integer in the previous row.
Thus, the largest integer in any row is six times the row number.
Therefore, the largest integer in row 30 is $6 \times 30 = 180$.

(b) By a similar argument to part (a), it follows that the largest integer in row 2012 is $6 \times 2012 = 12\,072$.
We find the other numbers in the row by counting backwards.
Thus, the six integers in row 2012 are $12\,072, 12\,071, 12\,070, 12\,069, 12\,068, 12\,067$.
The sum of the six integers in row 2012 is

$$12\,072 + 12\,071 + 12\,070 + 12\,069 + 12\,068 + 12\,067 = 72\,417.$$

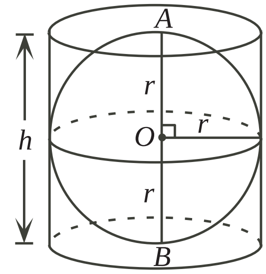
(c) Again from part (a), the largest integer in any row is six times the row number.
Thus to find the approximate row in which 5000 appears, we divide 5000 by 6.
Since $\frac{5000}{6} = 833\frac{1}{3}$, and $6 \times 833 = 4998$, then the largest integer in row 833 is 4998.
Therefore, row 834 contains the next six consecutive integers from 4999 to 5004.
(We can check this by recognizing that $6 \times 834 = 5004$.)
Thus, the integer 5000 appears in row 834.
Next, we recognize that all even numbered rows list the largest integer in the row beginning in column A through to the smallest integer in column F.
Since row 834 is an even numbered row, then the integers are listed in the order $5004, 5003, 5002, 5001, 5000, 4999$, with 5004 beginning in column A.
Therefore, the integer 5000 appears in row 834, column E.

(d) The largest integer in row r is $6 \times r$ or $6r$.
Since each row contains six consecutive integers, counting backwards the remaining five integers in the row are, $6r - 1, 6r - 2, 6r - 3, 6r - 4, 6r - 5$.
Thus, the sum of the six integers in row r is

$$6r + (6r - 1) + (6r - 2) + (6r - 3) + (6r - 4) + (6r - 5) = 36r - 15.$$

Since we require the sum of the six integers in the row to be greater than $10\,000$, then $36r - 15 > 10\,000$ or $36r > 10\,015$ or $r > \frac{10\,015}{36}$, and so $r > 278\frac{7}{36}$.
But the row number r must be a whole number, so $r \geq 279$.
Since we also require the sum of the six integers in the row to be less than $20\,000$, then $36r - 15 < 20\,000$ or $36r < 20\,015$ or $r < \frac{20\,015}{36}$, and so $r < 555\frac{35}{36}$.
But the row number r must be a whole number, so $r \leq 555$.
Therefore, the rows in which the six integers have a sum greater than $10\,000$ and less than $20\,000$ are $279, 280, 281, \ldots, 555$.
This gives $555 - 279 + 1$ or 277 rows that satisfy the requirements.

4. (a) The point A lies on the sphere, vertically above the centre of the sphere, O.
Similarly, point B lies on the sphere, vertically below the centre of the sphere.
The top and bottom faces of the cylinder touch the sphere at points A and B respectively, as shown.
The segment AB passes through the centre of the sphere.
Since OA is a radius of the sphere, it has length r.
Similarly, OB has length r.
Thus, the length of segment AB is $2r$.
However, segment AB also represents the perpendicular distance between the top and bottom faces of the cylinder and thus has length equal to the height of the cylinder, h.
Therefore, the equation relating the height of the cylinder to the radius of the sphere is $h = 2r$.



(b) The volume of the sphere is given by the formula $\frac{4}{3}\pi r^3$.
Since the volume of the sphere is 288π, then $\frac{4}{3}\pi r^3 = 288\pi$ or $4\pi r^3 = 3 \times 288\pi$, and so $r^3 = \frac{3\times 288\pi}{4\pi} = 216$.
The volume of the cylinder is given by the formula $\pi r^2 h$.
From part (a), $h = 2r$.
Thus, the volume of the cylinder is $\pi r^2 h = \pi r^2(2r) = 2\pi r^3$.
Since $r^3 = 216$, then the volume of the cylinder is $2\pi(216) = 432\pi$.

(c) The shape of the space that Darla is able to travel within is determined by the set of points that are exactly 1 km from the nearest point on the cube.
The surface of the cube is comprised of three types of points.
These are points on a face of the cube, points on an edge of the cube, and points at a vertex of the cube.
To determine the volume of space that Darla is able to occupy, we will consider each of these three types of surface points as separate cases.

Case 1 - Points on a face of the cube.
The question we must answer is, "What set of points is exactly 1 km away from the nearest point that is on a face of the cube?"
Consider that if Darla begins from any point on a face of the cube, the maximum distance that she can travel is 1 km.
To travel 1 km away from that point, but not be nearer to any other point on the cube, Darla must travel in a direction perpendicular to the face of the cube.
If Darla travels 1 km in a direction perpendicular to the face of a cube, beginning from each of the points on a face of the cube, then the shape of this space that she can occupy is another cube of side length 1 km.
This new cube extends directly outward from the original cube.
Since this can be repeated from each of the 6 faces of the original cube, then in this case Darla can occupy a volume of space equal to $6 \times 1 \times 1 \times 1$ or 6 km^3, as shown in Figure 1.

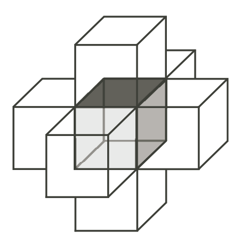


Figure 1

Case 2 - Points on an edge of the cube.

The question we must answer is, "What set of points is exactly 1 km away from the nearest point that is on an edge of the cube?"
Consider point A, the midpoint of the edge on which it lies.
Let points B and C be the midpoints of their respective edges also, as shown in Figure 2.
Darla can travel to both points B and C since point A is on the original cube, 1 km away from each of these points.
However, Darla can also travel from A to any point on the arc BC. This arc is one quarter of the circumference of the circle with centre A, radius 1 km, and passing through points B and C (since $\angle BAC = 90°$).
Darla can repeat this movement, beginning from any point on this edge. Thus, this shape of space that can be occupied is one quarter of a cylinder of radius 1 km and height 1 km, and has a volume of $\frac{1}{4}\pi(1)^2(1) = \frac{1}{4}\pi$ km^3 (see Figure 3).
Since this can be repeated from any point on each of the 12 edges of the original cube, then in this case Darla can occupy a volume of space equal to $12 \times \frac{1}{4}\pi$ or 3π km^3, as shown in Figure 4.

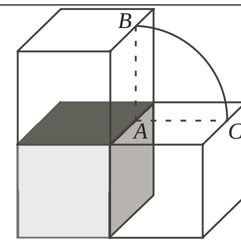


Figure 2

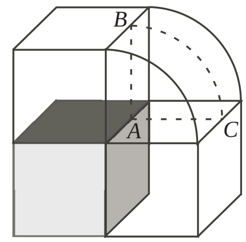


Figure 3

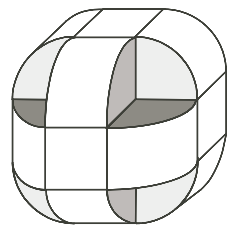


Figure 4

Case 3 - Points at a vertex of the cube.

The question we must answer is, "What set of points is exactly 1 km away from the nearest point that is at a vertex of the cube?" Consider point P, a vertex of the original cube.
Let points Q, R and S be vertices of external cubes, as shown in Figure 5.
Darla can travel to points Q, R and S since point P is on the original cube, 1 km away from each of these points.
From Case 2, we also know that Darla can travel anywhere along the arcs QR, RS and SQ.
However, Darla can also travel up to 1 km outward from P to any point on the 3-dimensional surface contained within these 3 arcs.
Since $\angle SPQ = \angle SPR = \angle QPR = 90°$, this surface is one eighth of the surface of the sphere with centre P and radius 1 km (see Figure 5).
Thus, the volume of the space that can be occupied is $\frac{1}{8} \times \frac{4}{3}\pi(1)^3 = \frac{1}{6}\pi$ km^3.
Since this can be repeated from each of the 8 vertices of the original cube, then in this case Darla can occupy a volume of space equal to $8 \times \frac{1}{6}\pi$ or $\frac{4}{3}\pi$ km^3, as shown in Figure 6.

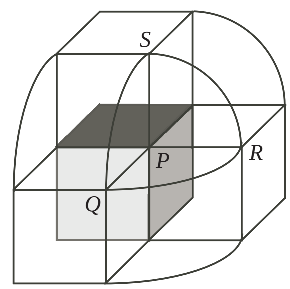


Figure 5

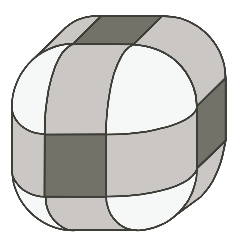


Figure 6

The total volume of space that Darla can occupy is the sum of the volume of space given by the 3 cases above.
That is, Darla can occupy a volume of space equal to $6 + 3\pi + \frac{4}{3}\pi$ or $(6 + \frac{13\pi}{3})$ km^3.