



***The CENTRE for EDUCATION in
MATHEMATICS and COMPUTING***

2011 Galois Contest

Wednesday, April 13, 2011

Solutions

1. (a) Using Jackson's rule, the second term of Fabien's sequence is $\frac{1}{1-2} = \frac{1}{-1} = -1$.
- (b) Since the second term is -1 , the third term is $\frac{1}{1-(-1)} = \frac{1}{1+1} = \frac{1}{2}$.
- Since the third term is $\frac{1}{2}$, the fourth term is $\frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$.
- Since the fourth term is 2 , the fifth term is $\frac{1}{1-2} = \frac{1}{-1} = -1$.
- (c) Since the fourth term, 2 , is equal to the first term and each term depends only on the previous term, then the sequence of terms repeats every 3 terms.
- That is, the sequence of numbers produced is $2, -1, \frac{1}{2}, 2, -1, \frac{1}{2}, 2, \dots$
- Since the terms of the sequence $2, -1, \frac{1}{2}$ repeat every three terms, then we must determine how many groups of three terms there are in the first 2011 terms.
- Since $2011 = 670 \times 3 + 1$, the sequence $2, -1, \frac{1}{2}$ repeats 670 times (giving the first 2010 terms), with the 2011th term being 2 .
- That is, there are 671 terms equal to 2 in Fabien's sequence.
- (d) The repeating cycle identified in part (c) has a sum of $2 + (-1) + \frac{1}{2} = \frac{3}{2}$.
- This complete cycle repeats 670 times.
- Thus, the sum of the first 2010 terms in the sequence is $670 \times \frac{3}{2} = 1005$.
- Since the 2011th term is 2 , the sum of all terms in Fabien's sequence is $1005 + 2$ or 1007 .
2. (a) We organize the possibilities that may appear on the coins in the table below.

'5 coin'	'7 coin'	'10 coin'	Score
0	0	0	0
5	0	0	5
0	7	0	7
0	0	10	10
5	7	0	12
5	0	10	15
0	7	10	17
5	7	10	22

The other possible scores are $0, 5, 7, 10, 12, 15$, and 22 .

(b) *Solution 1*

Since the three given scores are different from one another, a different coin must be showing a 0 on each of the three tosses.

That is, after the three tosses each coin has had its zero side appear once, and its non-zero side appear twice.

This means that the total of the scores from all three tosses, $60 + 110 + 130 = 300$, represents twice the sum of the number on the non-zero sides of the three coins.

If twice the sum of the non-zero numbers on the three coins equals 300, then the sum of the non-zero numbers on the three coins is $300 \div 2$ or 150.

Since the maximum possible score occurs when the non-zero number appears on each of the three coins, then the maximum possible score is 150.

Solution 2

Since the three given scores are different from one another, a different coin must be showing a 0 on each of the three tosses.

Let the non-zero number appearing on each of the three coins be a , b and c .

Since exactly one of the three coins shows a zero on each of the three tosses, we may assume without loss of generality that $a + b = 60$, $a + c = 110$, and $b + c = 130$.

Adding the left sides of these three equations gives $a + b + a + c + b + c$ or $2a + 2b + 2c$.

Adding the right sides of the three equations gives $60 + 110 + 130$ or 300 .

Since $2a + 2b + 2c = 300$, then $2(a + b + c) = 300$ and so $a + b + c = 150$.

This sum, 150, represents the score when the non-zero number appears on each of the three coins.

Since the maximum possible score occurs when the non-zero number appears on each of the three coins, then the maximum possible score is 150.

- (c) We organize the possibilities that may appear on the third coin in the table below, accounting for all of the possible combinations of values from the first two coins:

Appearing on the '25 coin'	Appearing on the '50 coin'	Appearing on the 3 rd coin
0	0	$170 - 0 = 170$
25	0	$170 - 25 = 145$
0	50	$170 - 50 = 120$
25	50	$170 - 75 = 95$

The possible non-zero numbers that may appear on the third coin are 170, 145, 120, and 95.

3. (a) Since $\angle ABP = 90^\circ$, $\triangle ABP$ is a right-angled triangle.
 By the Pythagorean Theorem, $BP^2 = AP^2 - AB^2$ or $BP^2 = 20^2 - 16^2$ or $BP^2 = 144$ and so $BP = 12$, since $BP > 0$.
 Since $\angle QTP = 90^\circ$, $\triangle QTP$ is a right-angled triangle with $PT = 12$.
 Since $PT = BP = 12$, then by the Pythagorean Theorem, $QT^2 = QP^2 - PT^2$ or $QT^2 = 15^2 - 12^2$ or $QT^2 = 81$ and so $QT = 9$, since $QT > 0$.
- (b) In triangles PQT and DQS , $\angle PTQ = \angle DSQ = 90^\circ$.
 Also, $\angle PQT$ and $\angle DQS$ are vertically opposite angles and are therefore equal.
 Since $\angle PTQ = \angle DSQ$, $\angle PQT = \angle DQS$, and the sum of the 3 angles in any triangle is 180° , then the third pair of corresponding angles, $\angle QPT$ and $\angle QDS$, are also equal.
 Since the corresponding angles in these two triangles are equal, then $\triangle PQT$ and $\triangle DQS$ are similar triangles.
- (c) Since $ABCD$ is a rectangle and TS is perpendicular to BC , then $ABTS$ is also a rectangle.
 Thus, $TS = BA = 16$ and $QS = TS - QT = 16 - 9 = 7$.
 As shown in part (b), $\triangle PQT$ and $\triangle DQS$ are similar triangles.
 Therefore, the ratios of corresponding side lengths in these two triangles are equal.
 That is, $\frac{SD}{TP} = \frac{QS}{QT}$ or $\frac{SD}{12} = \frac{7}{9}$ or $SD = 12 \times \frac{7}{9} = \frac{28}{3}$.
- (d) *Solution 1*
 In $\triangle QAS$ and $\triangle RAD$, $\angle QAS$ and $\angle RAD$ are common (the same) angles and thus are equal.
 Since $ABCD$ is a rectangle, $\angle RDA = 90^\circ = \angle QSA$.
 Since $\angle QAS = \angle RAD$, $\angle RDA = \angle QSA$, and the sum of the 3 angles in any triangle is 180° , then the third pair of corresponding angles, $\angle SQA$ and $\angle DRA$, are also equal.
 Since the corresponding angles in these two triangles are equal, then $\triangle QAS$ and $\triangle RAD$ are similar triangles.

Therefore, the ratios of corresponding side lengths in these two triangles are equal.

That is, $\frac{RD}{QS} = \frac{DA}{SA}$ or $RD = QS \times \frac{DA}{SA}$.

However, $DA = AS + SD = 24 + \frac{28}{3} = \frac{100}{3}$, and so $RD = 7 \times \frac{(\frac{100}{3})}{24} = 7 \times \frac{100}{72}$ or $RD = \frac{175}{18}$.

Since $\triangle QAS$ and $\triangle RAD$ are similar triangles, then $\frac{RA}{QA} = \frac{RD}{QS}$.

Thus, $RA = QA \times \frac{RD}{QS} = 25 \times \frac{(\frac{175}{18})}{7}$ or $RA = 25 \times \frac{25}{18}$, and so $RA = \frac{625}{18}$.

Since $QR = RA - QA$, then $QR = \frac{625}{18} - 25$ or $QR = \frac{625 - 450}{18}$, and so $QR = \frac{175}{18}$.

Therefore, $QR = RD$.

Solution 2

In triangles PQA and TQP , the ratios of corresponding side lengths are equal.

That is, $\frac{PA}{TP} = \frac{PQ}{TQ} = \frac{QA}{QP}$ or $\frac{20}{12} = \frac{15}{9} = \frac{25}{15} = \frac{5}{3}$.

Therefore, $\triangle PQA$ and $\triangle TQP$ are similar triangles and thus their corresponding angles are equal.

That is, $\angle PQA = \angle TQP = \alpha$.

Since $\angle RQD$ and $\angle PQA$ are vertically opposite angles, then $\angle RQD = \angle PQA = \alpha$.

Since CD and TS are parallel, then by the Parallel Lines Theorem $\angle RDQ = \angle TQP = \alpha$.

Therefore, $\angle RDQ = \angle RQD$ and so $\triangle RQD$ is an isosceles triangle with $QR = RD$.

4. (a) Since $T(4) = 10$ and $T(10) = 55$, then $T(a) = T(10) - T(4) = 45$.

That is, $\frac{a(a+1)}{2} = 45$ or $a^2 + a = 90$, and so $a^2 + a - 90 = 0$.

Since $a > 0$ and $(a-9)(a+10) = 0$, then $a = 9$.

- (b) The left side of the equation, $T(b+1) - T(b)$, gives $\frac{(b+1)(b+2)}{2} - \frac{b(b+1)}{2}$, which simplifies to $\frac{b^2 + 3b + 2 - b^2 - b}{2}$ or $\frac{2b+2}{2}$ or $b+1$.

That is, $b+1$ is equal to $T(x)$, a triangular number.

Since $b > 2011$, we are looking for the the smallest triangular number greater than 2012.

After some trial and error, we observe that $T(62) = 1953$ and $T(63) = 2016$, and so $b+1 = 2016$ or $b = 2015$ is the smallest value that works.

- (c) Since $T(28) = 406$, the second equation gives $c + d + e = 406$ or $e = 406 - (c + d)$.

Next, we simplify the first equation.

$$\begin{aligned} T(c) + T(d) &= T(e) \\ \frac{c(c+1)}{2} + \frac{d(d+1)}{2} &= \frac{e(e+1)}{2} \\ c(c+1) + d(d+1) &= e(e+1) \end{aligned}$$

We now substitute $e = 406 - (c + d)$ into this equation above and simplify.

$$\begin{aligned} c(c+1) + d(d+1) &= e(e+1) \\ c(c+1) + d(d+1) &= (406 - (c+d))(407 - (c+d)) \\ c^2 + c + d^2 + d &= 406 \times 407 - 406(c+d) - 407(c+d) + (c+d)^2 \\ c^2 + c + d^2 + d &= 406 \times 407 - 813(c+d) + (c+d)^2 \\ c^2 + c + d^2 + d &= 406 \times 407 - 813(c+d) + c^2 + 2cd + d^2 \\ c+d &= 406 \times 407 - 813(c+d) + 2cd \\ 2cd &= c+d + 813(c+d) - 406 \times 407 \\ 2cd &= 814(c+d) - 406 \times 407 \\ cd &= 407(c+d) - 203 \times 407 \\ cd &= 407(c+d - 203), \end{aligned}$$

as required.

(d) *Solution 1*

Using the result from part (c), we are looking to find all triples (c, d, e) of positive integers, where $c \leq d \leq e$, such that $cd = 407(c + d - 203)$.

Since the right side of this equation is divisible by 407, then the left side must also be divisible by 407.

Observe that $407 = 37 \times 11$.

Since cd is divisible by 407 and 407 is divisible by 37, then cd is divisible by 37.

Since 37 is a prime number, then one of c or d must be divisible by 37.

Since $c + d + e = 406$ then $d + e \leq 406$.

Since $d \leq e$, then $d + d \leq 406$ or $d \leq 203$.

Therefore, $c \leq d \leq 203$.

Thus, one of c or d is a multiple of 37 that is less than 203.

The largest multiple of 37 less than 203 is $5 \times 37 = 185$.

Next, we try the values $d = 37, 74, 111, 148, 185$ in the equation $cd = 407(c + d - 203)$ to see if we get an integer value for c .

The system of equations that we are solving is symmetric in c and d .

That is, exchanging c and d in the two equations yields the same two equations and thus the same solutions, but with c and d switched.

Therefore, if we happened to get a value of c larger than the value of d that we were trying, then we could just switch them.

In trying the possible values $d = 37, 74, 111, 148, 185$, we only obtain an integer value for c when $d = 185$.

The only triple (c, d, e) , where $c \leq d \leq e$, such that $cd = 407(c + d - 203)$ is $(33, 185, 188)$.

Solution 2

Using the result from part (c), we are looking to find all triples (c, d, e) of positive integers, where $c \leq d \leq e$, such that $cd = 407(c + d - 203)$.

Since the right side of this equation is divisible by 407, then the left side must also be divisible by 407.

Observe that $407 = 37 \times 11$.

Since cd is divisible by 407 and 407 is divisible by 37, then cd is divisible by 37.

Since 37 is a prime number, then one of c or d must be divisible by 37.

Suppose that d is divisible by 37, or that $d = 37n$ for some positive integer n .

(We will consider the possibility that it is c that is divisible by 37 later in the solution.)
 Since $c + d + e = 406$ and c, d, e are positive integers, then $1 \leq d \leq 404$ or $1 \leq n \leq 10$.
 With $d = 37n$ our equation $cd = 407(c + d - 203)$ becomes $37cn = 407(c + 37n - 203)$.
 Dividing through by 37, we get $cn = 11(c + 37n - 203)$ or $cn - 11c = 11 \times 37n - 11 \times 203$.
 Isolating c in this equation we have $c(n - 11) = 407n - 2233$ or $c = \frac{407n - 2233}{n - 11}$.

Since the numerator $407n - 2233$ can be written as $407n - 4477 + 2244$ or $407(n - 11) + 2244$,
 then we have $c = \frac{407(n - 11) + 2244}{n - 11}$ or $c = \frac{407(n - 11)}{n - 11} + \frac{2244}{n - 11}$ or $c = 407 + \frac{2244}{n - 11}$.

Since c is a positive integer, then $n - 11$ must divide 2244.

Since $1 \leq n \leq 10$, then $-10 \leq n - 11 \leq -1$.

Thus, the only possibilities for $n - 11$ are $-1, -2, -3, -4$, and -6 .

However, of these 5 possibilities only $n - 11 = -6$ gives a positive value for c .

Since $n - 11 = -6$, then $n = 5$, $d = 37 \times 5 = 185$, $c = 33$ and $e = 406 - (c + d) = 188$.

A triple (c, d, e) , where $c \leq d \leq e$, such that $cd = 407(c + d - 203)$ is $(33, 185, 188)$.

Earlier in this solution we made the assumption that d was divisible by 37.

Suppose that it is c that is divisible by 37 or that $c = 37n$ for some positive integer n .

Since $c + d + e = 406$ and c, d, e are positive integers, then $1 \leq c \leq 404$ or $1 \leq n \leq 10$.

With $c = 37n$ our equation $cd = 407(c + d - 203)$ becomes $37dn = 407(37n + d - 203)$.

Dividing through by 37, we get $dn = 11(37n + d - 203)$ or $dn - 11d = 11 \times 37n - 11 \times 203$.

Isolating d in this equation we have $d(n - 11) = 407n - 2233$ or $d = \frac{407n - 2233}{n - 11}$.

Since the numerator $407n - 2233$ can be written as $407n - 4477 + 2244$ or $407(n - 11) + 2244$,
 then we have $d = \frac{407(n - 11) + 2244}{n - 11}$ or $d = \frac{407(n - 11)}{n - 11} + \frac{2244}{n - 11}$ or $d = 407 + \frac{2244}{n - 11}$.

Since d is a positive integer, then $n - 11$ must divide 2244.

Since $1 \leq n \leq 10$, then $-10 \leq n - 11 \leq -1$.

Thus, the only possibilities for $n - 11$ are $-1, -2, -3, -4$, and -6 .

However, of these 5 possibilities only $n - 11 = -6$ gives a positive value for d .

Since $n - 11 = -6$, then $n = 5$, $c = 37 \times 5 = 185$, $d = 33$ and $e = 406 - (c + d) = 188$.

Since there is a restriction that $c \leq d \leq e$, then this solution is not possible.

The only triple (c, d, e) , where $c \leq d \leq e$, such that $cd = 407(c + d - 203)$ is $(33, 185, 188)$.