The CENTRE for EDUCATION
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2011 Canadian Senior Mathematics Contest

Tuesday, November 22, 2011
(in North America and South America)

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Solutions

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Part A

1. *Solution 1*

   Multiplying through, we obtain

$$2^4\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}\right)=16+\frac{16}{2}+\frac{16}{4}+\frac{16}{8}+\frac{16}{16}=16+8+4+2+1=31$$

   Solution 2

   Using a common denominator inside the parentheses, we obtain

$$2^4\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}\right)=16\left(\frac{16}{16}+\frac{8}{16}+\frac{4}{16}+\frac{2}{16}+\frac{1}{16}\right)=16\left(\frac{31}{16}\right)=31$$

   Answer: 31

2. Suppose that Daryl's age now is d and Joe's age now is j.
   Four years ago, Daryl's age was $d-4$ and Joe's age was $j-4$.
   In five years, Daryl's age will be $d+5$ and Joe's age will be $j+5$.
   From the first piece of given information, $d-4=3(j-4)$ and so $d-4=3j-12$ or $d=3j-8$.
   From the second piece of given information, $d+5=2(j+5)$ and so $d+5=2j+10$ or $d=2j+5$.
   Equating values of d, we obtain $3j-8=2j+5$ which gives $j=13$.
   Substituting, we obtain $d=2(13)+5=31$.
   Therefore, Daryl is 31 years old now.

   Answer: 31

3. When the red die is rolled, there are 6 equally likely outcomes. Similarly, when the blue die is rolled, there are 6 equally likely outcomes.
   Therefore, when the two dice are rolled, there are $6\times 6=36$ equally likely outcomes for the combination of the numbers on the top face of each. (These outcomes are Red 1 and Blue 1, Red 1 and Blue 2, Red 1 and Blue 3, ..., Red 6 and Blue 6.)
   The chart below shows these possibilities along with the sum of the numbers in each case:

		Blue Die					
		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Red	3	4	5	6	7	8	9
Die	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

   Since the only perfect squares between 2 and 12 are 4 (which equals 2^2) and 9 (which equals 3^2), then 7 of the 36 possible outcomes are perfect squares.
   Since each entry in the table is equally likely, then the probability that the sum is a perfect square is $\frac{7}{36}$.

   Answer: $\frac{7}{36}$

4. *Solution 1*

We find the prime factorization of 18 800:

$$18\,800 = 188 \cdot 100 = 2 \cdot 94 \cdot 10^2 = 2 \cdot 2 \cdot 47 \cdot (2 \cdot 5)^2 = 2^2 \cdot 47 \cdot 2^2 \cdot 5^2 = 2^4 5^2 47^1$$

If d is a positive integer divisor of 18 800, it cannot have more than 4 factors of 2, more than 2 factors of 5, more than 1 factor of 47, and cannot include any other prime factors. Therefore, if d is a positive integer divisor of 18 800, then $d = 2^a 5^b 47^c$ for some integers a, b and c with $0 \leq a \leq 4$ and $0 \leq b \leq 2$ and $0 \leq c \leq 1$.

Since we want to count all divisors d that are divisible by 235 and $235 = 5 \times 47$, then we need d to contain at least one factor of each of 5 and 47, and so $b \geq 1$ and $c \geq 1$. (Since $0 \leq c \leq 1$, then c must equal 1.)

Let D be a positive integer divisor of 18 800 that is divisible by 235.

Then D is of the form $d = 2^a 5^b 47^1$ for some integers a and b with $0 \leq a \leq 4$ and $1 \leq b \leq 2$.

Since there are 5 possible values for a and 2 possible values for b, then there are $5 \times 2 = 10$ possible values for D.

Therefore, there are 10 positive divisors of 18 800 that are divisible by 235.

Solution 2

Any positive divisor of 18 800 that is divisible by 235 is of the form $235q$ for some positive integer q. Thus, we want to count the number of positive integers q for which $235q$ divides exactly into 18 800.

For $235q$ to divide exactly into 18 800, we need $(235q)d = 18800$ for some positive integer d.

Simplifying, we want $qd = \frac{18800}{235} = 80$ for some positive integer d.

This means that we want to count the positive integers q for which there is a positive integer d such that $qd = 80$.

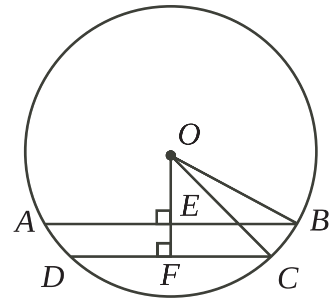
In other words, we want to count the positive divisors of 80.

We can do this using a similar method to that in (a), or since 80 is relatively small, we can list the divisors: $1, 2, 4, 5, 8, 10, 16, 20, 40, 80$.

There are 10 such positive divisors, so 18 800 has 10 positive divisors that are divisible by 235.

Answer: 10

5. Since OF passes through the centre of the circle and is perpendicular to each of chord AB and chord DC, then it bisects each of AB and DC. (That is, $AE = EB$ and $DF = FC$.)
To see that $AE = EB$, we could join O to A and O to B. Since $OA = OB$ (as they are radii), OE is common to each of $\triangle OAE$ and $\triangle OBE$, and each of these triangles is right-angled, then the triangles are congruent and so $AE = EB$. Using a similar approach shows that $DF = FC$.
Since $AE = EB$ and $AB = 8$, then $AE = EB = 4$.
Since $DF = FC$ and $DC = 6$, then $DF = FC = 3$.
Join O to B and O to C.
Let r be the radius of the circle and let $OE = x$.
Since $\triangle OEB$ is right-angled with $OE = x$, $EB = 4$ and $OB = r$, then $r^2 = x^2 + 4^2$ by the Pythagorean Theorem.
Since $OE = x$ and $EF = 1$, then $OF = x + 1$.
Since $\triangle OFC$ is right-angled with $OF = x+1$, $FC = 3$ and $OC = r$, then $r^2 = (x+1)^2 + 3^2$ by the Pythagorean Theorem.
Subtracting the first equation from the second, we obtain $0 = (x^2 + 2x + 1 + 9) - (x^2 + 16)$ or $0 = 2x - 6$ or $x = 3$.
Since $x = 3$, then $r^2 = 3^2 + 4^2 = 25$ and since $r > 0$, we get $r = 5$.



Answer: 5

6. Let R_1, R_2 and R_3 represent the three rows, C_1, C_2 and C_3 the three columns, D_1 the diagonal from the bottom left to the top right, and D_2 the diagonal from the top left to the bottom right. Since the sum of the numbers in R_1 equals the sum of the numbers in D_1, then

$$\log a + \log b + \log x = \log z + \log y + \log x$$

Simplifying, we get $\log a + \log b = \log z + \log y$ and so $\log(ab) = \log(yz)$ or $ab = yz$.
Thus, $z = \dfrac{ab}{y}$.
Since the sum of the numbers in C_1 equals the sum of the numbers in R_2, then

$$\log a + p + \log z = p + \log y + \log c$$

Simplifying, we get $\log a + \log z = \log y + \log c$ and so $\log(az) = \log(cy)$ or $az = cy$.
Thus, $z = \dfrac{cy}{a}$.
Since $z = \dfrac{ab}{y}$ and $z = \dfrac{cy}{a}$, then we obtain $\dfrac{ab}{y} = \dfrac{cy}{a}$ or $y^2 = \dfrac{a^2b}{c}$.
Since $a, b, c, y > 0$, then $y = \dfrac{ab^{1/2}}{c^{1/2}}$.
Since the sum of the numbers in C_3 equals the sum of the numbers in D_2, then

$$\log x + \log c + r = \log a + \log y + r$$

Simplifying, we get $\log x + \log c = \log a + \log y$ and so $\log(xc) = \log(ay)$ or $xc = ay$.
Thus, $x = \dfrac{ay}{c}$.
Therefore, $xyz = \dfrac{ay}{c} \cdot y \cdot \dfrac{cy}{a} = y^3 = \left(\dfrac{ab^{1/2}}{c^{1/2}}\right)^3 = \dfrac{a^3b^{3/2}}{c^{3/2}}$.
(Note that there are many other ways to obtain this same answer.)

Answer: $xyz = \dfrac{a^3b^{3/2}}{c^{3/2}}$

Part B

1. (a) The points A and B are the points where the parabola with equation $y = 25 - x^2$ intersects the x-axis.
To find their coordinates, we solve the equation $0 = 25 - x^2$ to get $x^2 = 25$ or $x = \pm 5$.
Thus, A has coordinates $(-5, 0)$ and B has coordinates $(5, 0)$.
Therefore, $AB = 5 - (-5) = 10$.

(b) Since $ABCD$ is a rectangle, $BC = AD$ and $\angle DAB = 90^\circ$.
Since $BD = 26$ and $AB = 10$, then by the Pythagorean Theorem,

$$AD = \sqrt{BD^2 - AB^2} = \sqrt{26^2 - 10^2} = \sqrt{676 - 100} = \sqrt{576} = 24$$

since $AD > 0$.
Since $BC = AD$, then $BC = 24$.

(c) Since $ABCD$ is a rectangle with sides parallel to the axes, then D and C are vertically below A and B, respectively.
Since $AD = BC = 24$, A has coordinates $(-5, 0)$ and B has coordinates $(5, 0)$, then D has coordinates $(-5, -24)$ and C has coordinates $(5, -24)$.
Thus, line segment DC lies along the line with equation $y = -24$.
Therefore, the points E and F are the points of intersection of the line $y = -24$ with the parabola with equation $y = 25 - x^2$.
To find their coordinates, we solve $-24 = 25 - x^2$ to get $x^2 = 49$ or $x = \pm 7$.
Thus, E and F have coordinates $(-7, -24)$ and $(7, -24)$ and so $EF = 7 - (-7) = 14$.

2. (a) If x and y are positive integers with $\dfrac{2x + 11y}{3x + 4y} = 1$, then $2x + 11y = 3x + 4y$ or $7y = x$.
We try $x = 7$ and $y = 1$.
In this case, $\dfrac{2x + 11y}{3x + 4y} = \dfrac{2(7) + 11(1)}{3(7) + 4(1)} = \dfrac{25}{25} = 1$, as required.
Therefore, the integers $x = 7$ and $y = 1$ have the required property.
(In fact, any pair of positive integers (x, y) with $x = 7y$ will have the required property.)

(b) Suppose $u = \dfrac{a}{b}$ and $v = \dfrac{c}{d}$ for some positive integers a, b, c, d.
The average of u and v is $\dfrac{1}{2}(u + v) = \dfrac{1}{2}\left(\dfrac{a}{b} + \dfrac{c}{d}\right) = \dfrac{1}{2}\left(\dfrac{ad + bc}{bd}\right) = \dfrac{ad + bc}{2bd}$.
Since $u = \dfrac{a}{b} = \dfrac{ax}{bx}$ and $v = \dfrac{c}{d} = \dfrac{cy}{dy}$ for all positive integers x and y, then each fraction of the form $\dfrac{ax + cy}{bx + dy}$ is a mediant of u and v.
Can we write $\dfrac{ad + bc}{2bd}$ in the form $\dfrac{ax + cy}{bx + dy}$ for some positive integers x and y?
Yes, we can. If $x = d$ and $y = b$, then $\dfrac{ax + cy}{bx + dy} = \dfrac{ad + cb}{bd + db} = \dfrac{ad + bc}{2bd}$.
Thus, writing $u = \dfrac{ad}{bd}$ and $v = \dfrac{bc}{bd}$ gives us the mediant $\dfrac{ad + bc}{bd + bd} = \dfrac{ad + bc}{2bd}$, which equals the average of u and v.
Therefore, the average of u and v is indeed a mediant of u and v.

(c) Suppose that u and v are two positive rational numbers with $u < v$.
Any mediant m of u and v is of the form $\dfrac{a+c}{b+d}$ where $u = \dfrac{a}{b}$ and $v = \dfrac{c}{d}$ for some positive integers a, b, c, d.
Since $u < v$, then $\dfrac{a}{b} < \dfrac{c}{d}$ and so $ad < bc$ (since $b, d > 0$).
We need to show that $u < m$ and that $m < v$.
To do this, we show that $m - u > 0$ and that $v - m > 0$.
Consider $m - u$:

$$m - u = \frac{a+c}{b+d} - \frac{a}{b} = \frac{b(a+c) - a(b+d)}{b(b+d)} = \frac{ab + bc - ab - ad}{b(b+d)} = \frac{bc - ad}{b(b+d)}$$

Since $a, b, c, d > 0$, then the denominator of this fraction is positive. Since $bc > ad$, then the numerator of this fraction is positive.
Therefore, $m - u = \dfrac{bc - ad}{b(b+d)} > 0$, so $m > u$.
Consider $v - m$:

$$v - m = \frac{c}{d} - \frac{a+c}{b+d} = \frac{c(b+d) - d(a+c)}{d(b+d)} = \frac{bc + cd - ad - cd}{d(b+d)} = \frac{bc - ad}{d(b+d)}$$

Since $a, b, c, d > 0$, then the denominator of this fraction is positive. Since $bc > ad$, then the numerator of this fraction is positive.
Therefore, $v - m = \dfrac{bc - ad}{d(b+d)} > 0$, so $v > m$.
Thus, $u < m < v$, as required.

3. (a) We list all of the possible products by starting with all of those beginning with a_1 (that is, with $i = 1$), then all of those beginning with a_2, then all of those beginning with a_3:

$$a_1a_2a_3 = (-1)\cdot(-1)\cdot 1 = 1$$
$$a_1a_2a_4 = (-1)\cdot(-1)\cdot 1 = 1$$
$$a_1a_2a_5 = (-1)\cdot(-1)\cdot 1 = 1$$
$$a_1a_3a_4 = (-1)\cdot 1\cdot 1 = -1$$
$$a_1a_3a_5 = (-1)\cdot 1\cdot 1 = -1$$
$$a_1a_4a_5 = (-1)\cdot 1\cdot 1 = -1$$
$$a_2a_3a_4 = (-1)\cdot 1\cdot 1 = -1$$
$$a_2a_3a_5 = (-1)\cdot 1\cdot 1 = -1$$
$$a_2a_4a_5 = (-1)\cdot 1\cdot 1 = -1$$
$$a_3a_4a_5 = 1\cdot 1\cdot 1 = 1$$

Of the ten products, 4 are equal to 1.

(b) Each product $a_ia_ja_k$ is equal to 1 or -1, depending on whether it includes an even number of factors of -1 or an odd number of factors of -1.
If $a_ia_ja_k$ includes three 1s and zero (-1)s, it equals 1.
If $a_ia_ja_k$ includes two 1s and one (-1), it equals -1.
If $a_ia_ja_k$ includes one 1 and two (-1)s, it equals 1.
If $a_ia_ja_k$ includes zero 1s and three (-1)s, it equals -1.
Since the sequence includes m terms equal to -1 and p terms equal to 1, then

- the number of ways of choosing three 1s and zero (-1)s is $\dbinom{p}{3}\dbinom{m}{0}$,
- the number of ways of choosing two 1s and one (-1) is $\dbinom{p}{2}\dbinom{m}{1}$,
- the number of ways of choosing one 1 and two (-1)s is $\dbinom{p}{1}\dbinom{m}{2}$, and

- the number of ways of choosing zero 1s and three (-1)s is $\binom{p}{0}\binom{m}{3}$.

Therefore, the number of products $a_i a_j a_k$ equal to 1 is $\binom{p}{3}\binom{m}{0}+\binom{p}{1}\binom{m}{2}$ and the number of products equal to -1 is $\binom{p}{2}\binom{m}{1}+\binom{p}{0}\binom{m}{3}$.

If exactly half of the products are equal to 1, then half are equal to -1, and so the number of products of each kind are equal.

This property is equivalent to the following equations:

$$\begin{aligned}
\binom{p}{3}\binom{m}{0}+\binom{p}{1}\binom{m}{2} &= \binom{p}{2}\binom{m}{1}+\binom{p}{0}\binom{m}{3} \\
\frac{p(p-1)(p-2)}{3(2)(1)}\cdot 1 \;+\; p\cdot\frac{m(m-1)}{2(1)} &= \frac{p(p-1)}{2(1)}\cdot m \;+\; 1\cdot\frac{m(m-1)(m-2)}{3(2)(1)} \\
(p^3-3p^2+2p)+3pm(m-1) &= 3mp(p-1)+(m^3-3m^2+2m) \\
p^3-3p^2+2p+3m^2p-3mp &= 3mp^2-3mp+m^3-3m^2+2m
\end{aligned}$$

Each step so far is reversible so this last equation is equivalent to the desired property.

Grouping all terms on the left side and factoring, we obtain

$$\begin{aligned}
p^3-m^3-3(p^2-m^2)+2(p-m)+3m^2p-3mp^2 &= 0 \\
(p-m)(p^2+mp+m^2)-3(p-m)(p+m)+2(p-m)-3mp(p-m) &= 0 \\
(p-m)(p^2+mp+m^2-3(p+m)+2-3mp) &= 0 \\
(p-m)(p^2-3p-2mp+m^2-3m+2) &= 0
\end{aligned}$$

(We have used $p^3-m^3=(p-m)(p^2+mp+m^2)$ and $p^2-m^2=(p-m)(p+m)$.)

Therefore, the desired property is equivalent to the condition that either $p-m=0$ or $p^2-3p-2mp+m^2-3m+2=0$.

We count the number of pairs (m,p) in each of these two cases. The first case is easier than the second.

Case 1: $p-m=0$

We want to count the number of pairs (m,p) of positive integers that satisfy

$$1\leq m\leq p\leq 1000 \quad\text{and}\quad m+p\geq 3 \quad\text{and}\quad p-m=0$$

If $p-m=0$, then $p=m$. Since $1\leq m\leq p\leq 1000$ and $m+p\geq 3$, then the possible pairs (m,p) are of the form $(m,p)=(k,k)$ with k a positive integer ranging from $k=2$ to $k=1000$, inclusive. There are 999 such pairs.

Case 2: $p^2-3p-2mp+m^2-3m+2=0$

We want to count the number of pairs (m,p) of positive integers that satisfy

$$1\leq m\leq p\leq 1000 \quad\text{and}\quad m+p\geq 3 \quad\text{and}\quad p^2-3p-2mp+m^2-3m+2=0$$

We start with this last equation. We rewrite it as a quadratic equation in p (with coefficients in terms of m):

$$p^2-p(2m+3)+(m^2-3m+2)=0$$

By the quadratic formula, this equation is true if and only if

$$\begin{aligned} p &= \frac{(2m+3) \pm \sqrt{(2m+3)^2 - 4(m^2-3m+2)}}{2} \\ &= \frac{(2m+3) \pm \sqrt{(4m^2+12m+9)-(4m^2-12m+8)}}{2} \\ &= \frac{(2m+3) \pm \sqrt{24m+1}}{2} \end{aligned}$$

Since $m \geq 1$, then $24m+1 \geq 25$ and so $\sqrt{24m+1} \geq 5$.
This means that $\dfrac{(2m+3)-\sqrt{24m+1}}{2} \leq \dfrac{(2m+3)-5}{2} = m-1$.
In other words, if $p = \dfrac{(2m+3)-\sqrt{24m+1}}{2}$, then $p \leq m-1$. But $p \geq m$, so this is impossible.
Therefore, in Case 2 we are looking for pairs (m,p) of positive integers that satisfy

$$\text{(I) } 1 \leq m \leq p \leq 1000 \quad \text{and} \quad \text{(II) } m+p \geq 3 \quad \text{and} \quad \text{(III) } p = \frac{(2m+3)+\sqrt{24m+1}}{2}$$

From (III), for p to be an integer, it is necessary that $\sqrt{24m+1}$ be an integer (that is, for $24m+1$ to be a perfect square).
Since $24m+1$ is always an odd integer, then if $24m+1$ is a perfect square, it is an odd perfect square.
Since $24m+1$ is one more than a multiple of 3 (because $24m$ is a multiple of 3), then $24m+1$ is not a multiple of 3.
Therefore, if m gives an integer value for p, then $24m+1$ is a perfect square that is not divisible by 3.
So the question remains: Which odd perfect squares that are not divisible by 3 are of the form $24m+1$?
In fact, every odd perfect square that is not a multiple of 3 is of the form $24m+1$. (We will prove this fact at the very end of the solution.)
Therefore, the possible values of $24m+1$ are all odd perfect squares that are not multiples of 3. We will return to this.

We verify next that (II) is always true.
We can assume that $m \geq 1$. From the formula for p in terms of m, we can see that $p \geq \dfrac{(2(1)+3)+\sqrt{24(1)+1}}{2} = 5$, and so $m+p \geq 1+5 = 6$, and so the restriction $m+p \geq 3$ is true.

We verify next that part of (I) is always true.
Note that $p = \dfrac{(2m+3)+\sqrt{24m+1}}{2} \geq \dfrac{2m}{2} = m$, so the restriction $p \geq m$ is true.

Therefore, we want to count the pairs (m,p) of positive integers with $p \leq 1000$ and $p = \dfrac{(2m+3)+\sqrt{24m+1}}{2}$.

From above, the values of m that work are exactly those for which $24m+1$ is an odd perfect square that is not a multiple of 3.

We make a table of possible odd perfect square values for $24m + 1$ that are not multiples of 3, and the resulting values of m and of p (from the formula above):

$24m + 1$	m	p
$5^2 = 25$	1	5
$7^2 = 49$	2	7
$11^2 = 121$	5	12
$\vdots$	$\vdots$	$\vdots$
$143^2 = 20449$	852	925
$145^2 = 21025$	876	950
$147^2 = 22201$	925	1001

Since $p > 1000$ for this last row, we can stop. (Any larger value of $24m+1$ will give larger values of m and thus of p.)

We could have also solved the inequality $\dfrac{(2m+3)+\sqrt{24m+1}}{2} \leq 1000$ to obtain the restriction on m.

Finally, we need to count the pairs resulting from this table.
We do this by counting the number of odd perfect squares from 5^2 to 145^2 inclusive that are not multiples of 3.
This is equivalent to counting the number of odd integers from 5 to 145 that are not multiples of 3.
In total, there are 71 odd integers from 5 to 145 inclusive, since we can add 2 a total of 70 times starting from 5 to get 145.
The odd multiples of 3 between 5 and 145 are $9, 15, 21, \cdots, 135, 141$. There are 23 of these, since we can add 6 a total of 22 times starting from 9 to get 141.
Therefore, there are $71 - 23 = 48$ odd integers that are not multiples of 3 from 5 to 145 inclusive. This means that there are 48 pairs (m, p) in this case.

In total, there are then $999 + 48 = 1047$ pairs (m, p) that have the property that exactly half of the products $a_i a_j a_k$ are equal to 1.

Lastly, we need to prove the unproven fact from above:

Every odd perfect square that is not a multiple of 3 is of the form $24m + 1$

<u>Proof</u>
Suppose that k^2 is an odd perfect square that is not a multiple of 3.
Since k^2 is odd, then k is odd.
Since k^2 is not a multiple of 3, then k is not a multiple of 3.
Since k is odd, then it has one of the forms $k = 6q - 1$ or $k = 6q + 1$ or $k = 6q + 3$ for some integer q. (The form $k = 6q + 5$ is equivalent to the form $k = 6q - 1$.)
Since k is not a multiple of 3, then k cannot equal $6q + 3$ (which is $3(2q + 1)$).
Therefore, $k = 6q - 1$ or $k = 6q + 1$.
In the first case, $k^2 = (6q - 1)^2 = 36q^2 - 12q + 1 = 12(3q^2 - q) + 1$.
In the second case, $k^2 = (6q + 1)^2 = 36q^2 + 12q + 1 = 12(3q^2 + q) + 1$.
If q is an even integer, then $3q^2$ is even and so $3q^2 + q$ and $3q^2 - q$ are both even.

If q is an odd integer, then $3q^2$ is odd and so $3q^2 + q$ and $3q^2 - q$ are both even.
If $k^2 = 12(3q^2 + q) + 1$, then since $3q^2 + q$ is even, we can write $3q^2 + q = 2x$ for some integer x, and so $k^2 = 24x + 1$.
If $k^2 = 12(3q^2 - q) + 1$, then since $3q^2 - q$ is even, we can write $3q^2 - q = 2y$ for some integer y, and so $k^2 = 24y + 1$.
In either case, k^2 is one more than a multiple of 24, as required.