Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

Euclid Contest

Wednesday, April 7, 2010

STRONGER COMMUNITIES TOGETHER™

Time: $2\frac{1}{2}$ hours

Calculators are permitted, provided they are non-programmable and without graphic displays.

Do not open this booklet until instructed to do so. The paper consists of 10 questions, each worth 10 marks. Parts of each question can be of two types. **SHORT ANSWER** parts are worth 3 marks each. **FULL SOLUTION** parts are worth the remainder of the 10 marks for the question.

Instructions for SHORT ANSWER parts:

1. **SHORT ANSWER** parts are indicated like this: .
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

Instructions for FULL SOLUTION parts:

1. **FULL SOLUTION** parts are indicated like this: .
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, foolscap will be supplied by your supervising teacher. Insert these pages into your answer booklet. Be sure to write your name, school name and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution poorly presented will not earn full marks.

NOTE: At the completion of the Contest, insert the information sheet inside the answer booklet.

The names of some top-scoring students will be published in the Euclid Results on our Web site, http://www.cemc.uwaterloo.ca.

NOTES:	1.	Please read the instructions on the front cover of this booklet.
	2.	Write all answers in the answer booklet provided.
	3.	For questions marked " ", full marks will be given for a correct answer placed in the appropriate box in the answer booklet. **If an incorrect answer is given, marks may be given for work shown.** Students are strongly encouraged to show their work.
	4.	All calculations and answers should be expressed as exact numbers such as 4π, $2+\sqrt{7}$, etc., rather than as $12.566\ldots$ or $4.646\ldots$, except where otherwise indicated.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth, grade and sex, on the Student Information Form, and that you have answered the question about eligibility.

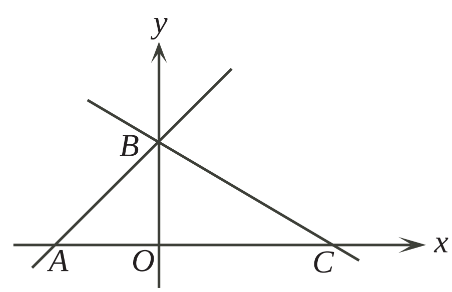
A Note about Writing Solutions

For each problem marked " ", a full solution is required. The solutions that you provide in the answer booklet should be well organized and contain mathematical statements and words of explanation when appropriate. Working out some of the details in rough on a separate piece of paper before writing your finished solution is a good idea. Your final solution should be written so that the marker can understand your approach to the problem and all of the mathematical steps of your solution.

1. (a) If $3^x = 27$, what is the value of 3^{x+2}?

   (b) If $2^5 3^{13} 5^9 x = 2^7 3^{14} 5^9$, what is the value of x?

   (c) Triangle ABC is enclosed by the lines $y = x + 2$, $y = -\frac{1}{2}x + 2$ and the x-axis. Determine the area of $\triangle ABC$.

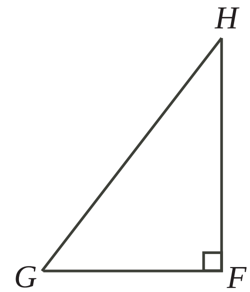


2. (a) Maria has a red package, a green package, and a blue package.
   The sum of the masses of the three packages is 60 kg.
   The sum of the masses of the red and green packages is 25 kg.
   The sum of the masses of the green and blue packages is 50 kg.
   What is the mass of the green package, in kg?

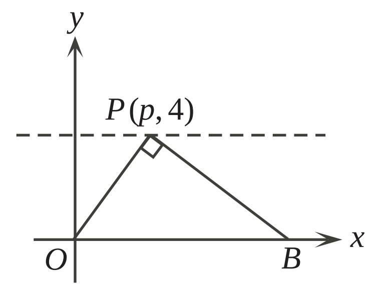
   (b) A *palindrome* is a positive integer that is the same when read forwards or backwards. For example, 151 is a palindrome. What is the largest palindrome less than 200 that is the sum of three consecutive integers?

   (c) If $(x + 1)(x - 1) = 8$, determine the numerical value of $(x^2 + x)(x^2 - x)$.

3. (a) Bea the bee sets out from her hive, H, and flies south for 1 hour to a field, F. She spends 30 minutes in the field, and then flies 45 minutes west to a garden, G. After spending 1 hour in the garden, she flies back to her hive along a straight line route. Bea always flies at the same constant speed. What is the total length of time, in minutes, that she is away from her hive?



(b) In the diagram, points $P(p, 4)$, $B(10, 0)$, and $O(0, 0)$ are shown. If $\triangle OPB$ is right-angled at P, determine all possible values of p.



4. (a) Thurka bought some stuffed goats and some toy helicopters. She paid a total of \$201. She did not buy partial goats or partial helicopters. Each stuffed goat cost \$19 and each toy helicopter cost \$17. How many of each did she buy?

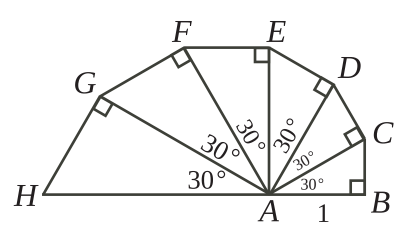
(b) Determine all real values of x for which $(x + 8)^4 = (2x + 16)^2$.

5. (a) If $f(x) = 2x + 1$ and $g(f(x)) = 4x^2 + 1$, determine an expression for $g(x)$.

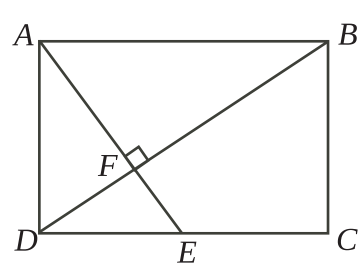
(b) A geometric sequence has 20 terms.
The sum of its first two terms is 40.
The sum of its first three terms is 76.
The sum of its first four terms is 130.
Determine how many of the terms in the sequence are integers.

(A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

6. (a) A snail's shell is formed from six triangular sections, as shown. Each triangle has interior angles of 30°, 60° and 90°. If AB has a length of 1 cm, what is the length of AH, in cm?



(b) In rectangle $ABCD$, point E is on side DC. Line segments AE and BD are perpendicular and intersect at F. If $AF = 4$ and $DF = 2$, determine the area of quadrilateral $BCEF$.

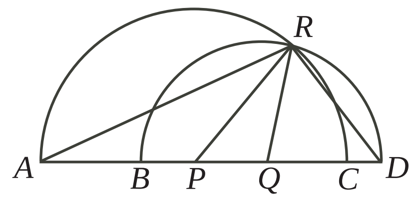


7. (a) Determine all real values of x for which $3^{(x-1)}9^{\frac{3}{2x^2}} = 27$.

(b) Determine all points (x, y) where the two curves $y = \log_{10}(x^4)$ and $y = (\log_{10} x)^3$ intersect.

8. (a) Oi-Lam tosses three fair coins and removes all of the coins that come up heads. George then tosses the coins that remain, if any. Determine the probability that George tosses exactly one head.

(b) In the diagram, points B, P, Q, and C lie on line segment AD. The semi-circle with diameter AC has centre P and the semi-circle with diameter BD has centre Q. The two semi-circles intersect at R. If $\angle PRQ = 40°$, determine the measure of $\angle ARD$.



9. (a) (i) If θ is an angle whose measure is not an integer multiple of $90°$, prove that

$$\cot\theta - \cot 2\theta = \frac{1}{\sin 2\theta}$$

(ii) Ross starts with an angle of measure $8°$ and doubles it 10 times until he obtains $8192°$. He then adds up the reciprocals of the sines of these 11 angles. That is, he calculates

$$S = \frac{1}{\sin 8°} + \frac{1}{\sin 16°} + \frac{1}{\sin 32°} + \cdots + \frac{1}{\sin 4096°} + \frac{1}{\sin 8192°}$$

Determine, without using a calculator, the measure of the acute angle α so that $S = \dfrac{1}{\sin\alpha}$.

(b) In $\triangle ABC$, $BC = a$, $AC = b$, $AB = c$, and $a < \frac{1}{2}(b + c)$.
Prove that $\angle BAC < \frac{1}{2}(\angle ABC + \angle ACB)$.

10. For each positive integer n, let $T(n)$ be the number of triangles with integer side lengths, positive area, and perimeter n. For example, $T(6) = 1$ since the only such triangle with a perimeter of 6 has side lengths 2, 2 and 2.

(a) Determine the values of $T(10)$, $T(11)$ and $T(12)$.

(b) If m is a positive integer with $m \geq 3$, prove that $T(2m) = T(2m - 3)$.

(c) Determine the smallest positive integer n such that $T(n) > 2010$.

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

For students...

Thank you for writing the 2010 Euclid Contest!
In 2009, more than 16 000 students from around the world registered to write the Euclid Contest.

Check out the CEMC's group on Facebook, called "Who is The Mathiest?".

If you are graduating from secondary school, good luck in your future endeavours!
If you will be returning to secondary school next year, encourage your teacher to register you for the 2010 Sun Life Financial Canadian Open Mathematics Challenge, which will be written in late November.
Visit our website

www.cemc.uwaterloo.ca

to find

- **More information about the Sun Life Financial Canadian Open Mathematics Challenge**
- **Free copies of past contests**
- **Workshops to help you prepare for future contests**
- **Information about our publications for mathematics enrichment and contest preparation**

For teachers...

Visit our website

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to

- **Obtain information about our 2010/2011 contests**
- **Learn about workshops and resources we offer for teachers**
- **Find your school results**