Canadian
Mathematics
Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Hypatia Contest

Wednesday, April 8, 2009

Solutions

1. Throughout this problem, we will need to know the total number of students in the class. According to the chart, the total number is $3+2+1+2+4+2+2+3+1=20$.

   (a) There are 20 students in total.
   Of these students, 2 have both green eyes and brown hair.
   Therefore, the percentage who have both green eyes and brown hair is $\frac{2}{20}\times 100\% = 10\%$.

   (b) There are 20 students in total.
   There are five entries in the table that appear in the either the "green eyes" row or the "brown hair" column. Note that the entry that is in both the "green eyes" row and the "brown hair" column is only counted once.
   The sum of these five entries is $2+4+2+3+2=13$.
   Therefore, the percentage of the students with either green eyes or brown hair is $\frac{13}{20}\times 100\% = 65\%$.

   (c) There are $2+4+2=8$ students in total with green eyes.
   Of these students, 2 have red hair.
   Therefore, the percentage of those with green eyes who have red hair is $\frac{2}{8}\times 100\% = 25\%$.

   (d) Initially, there are 20 students in the class, of whom $1+2+1=4$ have red hair.
   Suppose that x students with red hair join the class. There will then be $20+x$ students in total, of whom $4+x$ have red hair.
   We want $\dfrac{4+x}{20+x} = \dfrac{36}{100}$.
   Reducing the fraction on the right-hand side, this equation becomes $\dfrac{4+x}{20+x} = \dfrac{9}{25}$.
   Cross-multiplying, we obtain $25(4+x)=9(20+x)$ which gives $100+25x=180+9x$ and so $16x=80$ or $x=5$.
   Therefore, 5 students with red hair must join the class.
   (We could have seen that $x=5$ by inspection from the equation $\dfrac{4+x}{20+x} = \dfrac{9}{25}$ but this would not immediately guarantee us that this was the *only* solution to this equation.)

2. (a) *Solution 1*
   Suppose that the middle term of the sequence is x.
   If the common difference between terms in the sequence is d, then the first term is $x-d$ and the third term is $x+d$.
   Since the sum of the terms is 180, then $(x-d)+x+(x+d)=180$ or $3x=180$ and so $x=60$.
   Therefore, the middle term in the sequence is 60.

   Solution 2
   Suppose that the first term in the sequence is a and the common difference between terms in the sequence is d.
   Thus, the second term is $a+d$ and the third term is $(a+d)+d=a+2d$.
   The second term is the middle term, so we need to determine the value of $a+d$.
   Since the sum of the three terms is 180, then $a+(a+d)+(a+2d)=180$ or $3a+3d=180$ which gives $3(a+d)=180$.
   Thus, $a+d=60$ and so the middle term is 60.

   (b) *Solution 1*
   We show that the middle term (which is the third term) equals 36.

Suppose that the third term of the sequence is x.
If the common difference between terms in the sequence is d, then the second term is $x-d$, the first term is $x-2d$, the fourth term is $x+d$, and the fifth term is $x+2d$.
Since the sum of the terms is 180, then

$$\begin{aligned}(x-2d)+(x-d)+x+(x+d)+(x+2d) &= 180\\ 5x &= 180\\ x &= 36\end{aligned}$$

Therefore, the middle term in the sequence is 36.
(Note that if $d=0$ then all of the terms in the sequence are equal to 36, so it is possible for more than one term to equal 36.)

Solution 2
Suppose that the first term in the sequence is a and the common difference between terms in the sequence is d.
Thus, the second term is $a+d$, the third term is $(a+d)+d=a+2d$, the fourth term is $a+3d$, and the fifth term is $a+4d$.
The third term is the middle term, so we need to determine the value of $a+2d$.
Since the sum of the five terms is 180, then

$$\begin{aligned}a+(a+d)+(a+2d)+(a+3d)+(a+4d) &= 180\\ 5a+10d &= 180\\ 5(a+2d) &= 180\\ a+2d &= 36\end{aligned}$$

Thus, $a+2d=36$ and so the middle term equals 36.
(Note that if $d=0$ then all of the terms in the sequence are equal to 36, so it is possible for more than one term to equal 36.)

(c) Suppose that the first term in the sequence is a and the common difference between terms in the sequence is d.
Thus, the second term is $a+d$, the third term is $a+2d$, the fourth term is $a+3d$, the fifth term is $a+4d$, and the sixth term is $a+5d$.
We need to determine the sum of the first and sixth terms, which equals $a+(a+5d)=2a+5d$.
Since the sum of the six terms is 180, then

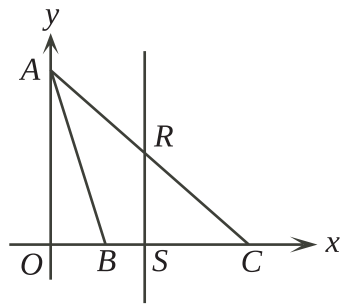
$$\begin{aligned}a+(a+d)+(a+2d)+(a+3d)+(a+4d)+(a+5d) &= 180\\ 6a+15d &= 180\\ 3(2a+5d) &= 180\\ 2a+5d &= 60\end{aligned}$$

Thus, $2a+5d=60$ and so the sum of the first and sixth terms is 60.

3. (a) The line through B that cuts the area of $\triangle ABC$ in half is the median – that is, the line through B and the midpoint M of AC.
This line cuts the area of the triangle in half, because if we consider AC as its base, then the height of each of $\triangle AMB$ and $\triangle CMB$ is equal to the distance of point B from the

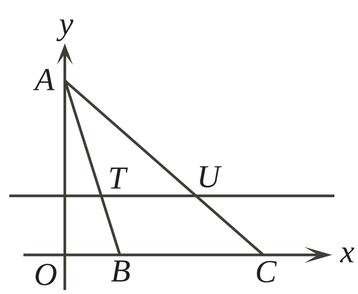
line through A and C. These two triangles also have equal bases because $AM = MC$, so their areas must be equal.
The midpoint, M, of AC has coordinates $\left(\frac{1}{2}(0+8), \frac{1}{2}(8+0)\right) = (4,4)$.
The slope of the line through $B(2,0)$ and $M(4,4)$ is $\dfrac{4-0}{4-2} = 2$.
Since this line passes through $B(2,0)$, it has equation $y - 0 = 2(x-2)$ or $y = 2x - 4$.

(b) Since line segment RS is vertical and S lies on BC, which is horizontal, then $\triangle RSC$ is right-angled at S.



Also, R lies on line segment AC, which has slope $\dfrac{0-8}{8-0} = -1$.
Since AC has a slope of -1, it makes an angle of 45° with the x-axis. In particular, the angle between RC and SC is 45°.
Since $\triangle RSC$ is right-angled at S and has a 45° angle at C, then the third-angle must be $180^\circ - 90^\circ - 45^\circ = 45^\circ$, which means that the triangle is right-angled and isosceles.
Suppose that $RS = SC = x$.
Since $\triangle RSC$ is right-angled, then the area of $\triangle RSC$ in terms of x is $\frac{1}{2}x^2$.
But we know that the area of $\triangle RSC$ is 12.5, so $\frac{1}{2}x^2 = 12.5$ or $x^2 = 25$.
Since $x > 0$, then $x = 5$.
This tells us that point S is 5 units to the left of C, so has coordinates $(8-5, 0) = (3,0)$.
Also, point R is 5 units above S, so has coordinates $(3, 0+5) = (3,5)$.

(c) *Solution 1*
Since line segment BC is horizontal and the line segment through T and U is also horizontal, then BC and TU are parallel.
Therefore $\angle ATU = \angle ABC$.



Since $\triangle ATU$ and $\triangle ABC$ also share a common angle at A, then $\triangle ATU$ is similar to $\triangle ABC$.
Since $\triangle ATU$ and $\triangle ABC$ are similar, then the ratio of their areas equals the square of the ratio of their heights.
Considering BC as the base of $\triangle ABC$, we see that its area is $\frac{1}{2}(8-2)(8) = 24$. Note that its height is 8 when considered in this direction.
Suppose that the height of $\triangle ATU$ considered from TU is h.

Then $\dfrac{13.5}{24} = \left(\dfrac{h}{8}\right)^2$ or $\dfrac{h^2}{64} = \dfrac{27}{48}$ or $h^2 = \dfrac{64(27)}{48} = 36$.
Since $h > 0$, then $h = 6$.
Therefore, the line segment TU is 6 units lower than the point $A(0,8)$, and so has equation $y = 8 - 6$ or $y = 2$.

Solution 2
Suppose that the equation of the horizontal line is $y = t$.
We find the coordinates of points T and U first.
To do this, we need to find the equation of the line through A and B and the equation of the line through A and C.
The line through A and B has slope $\dfrac{0-8}{2-0} = -4$ and passes through $(0,8)$, so has equation $y = -4x + 8$.
The line through A and C has slope $\dfrac{0-8}{8-0} = -1$ and passes through $(0,8)$, so has equation $y = -x + 8$.
The point T is the point on the line $y = -4x + 8$ with y-coordinate t.
To find the x-coordinate, we solve $t = -4x + 8$ to get $4x = 8 - t$ or $x = \frac{1}{4}(8-t)$.
The point U is the point on the line $y = -x + 8$ with y-coordinate t.
To find the x-coordinate, we solve $t = -x + 8$ to get $x = 8 - t$.
Therefore, T has coordinates $(\frac{1}{4}(8-t), t)$, U has coordinates $(8-t, t)$, and A is at $(0,8)$.
To find the area, we remember that TU is horizontal and has length $(8-t) - \frac{1}{4}(8-t) = \frac{3}{4}(8-t)$, and the distance from TU to A is $8 - t$.
Therefore, the area in terms of t is $\frac{1}{2}\left(\frac{3}{4}(8-t)\right)(8-t) = \frac{3}{8}(8-t)^2$.
Since we know that the area equals 13.5, then $\frac{3}{8}(8-t)^2 = 13.5$ or $(8-t)^2 = \frac{8}{3}(13.5) = 36$.
Note that $t < 8$ because line segment TU is below A, so $8 - t > 0$.
Therefore, $8 - t = 6$ and so $t = 8 - 6 = 2$.
Thus, the equation of the horizontal line through T and U is $y = 2$.

4. (a) Since $\triangle ABC$ is equilateral with side length 12 and X and Y are the midpoints of CA and CB, respectively, then $CX = CY = \frac{1}{2}(12) = 6$.
Since the height of the prism is 16 and Z is the midpoint of CD, then $CZ = \frac{1}{2}(16) = 8$.
Since faces $ACDE$ and $BCDF$ are rectangles, then $\angle ACD = \angle BCD = 90^\circ$.
Thus, $\triangle XCZ$ and $\triangle YCZ$ are right-angled at C.
By the Pythagorean Theorem, $XZ = \sqrt{CX^2 + CZ^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.
Similarly, $YZ = \sqrt{CY^2 + CZ^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.
Lastly, consider $\triangle CXY$.
We know that $CX = CY = 6$ and that $\angle XCY = 60^\circ$, because $\triangle ABC$ is equilateral.
Thus, $\triangle CXY$ is isosceles with $\angle CXY = \angle CYX$.
These angles must each be equal to $\frac{1}{2}(180^\circ - \angle XCY) = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
But this means that $\triangle CXY$ is equilateral, and so $XY = CX = CY = 6$.
Therefore, $XY = 6$ and $XZ = YZ = 10$.

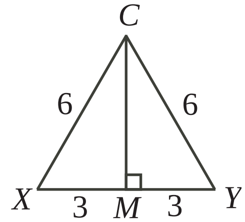
(b) To determine the surface area of solid $CXYZ$, we must determine the area of each of the four triangular faces.

Areas of $\triangle CZX$ and $\triangle CZY$
Each of these triangles is right-angled and has legs of lengths 6 and 8.
Therefore, the area of each is $\frac{1}{2}(6)(8) = 24$.

Area of $\triangle CXY$

This triangle is equilateral with side length 6.

We draw the altitude from C to M on XY. Since $\triangle CXY$ is equilateral, then M is the midpoint of XY.



Each of $\triangle CMX$ and $\triangle CMY$ is thus a 30°-60°-90° triangle, since each already has a 60° angle and a 90° angle.

Using the ratios from this special triangle, $CM = \frac{\sqrt{3}}{2}(CX) = \frac{\sqrt{3}}{2}(6) = 3\sqrt{3}$.

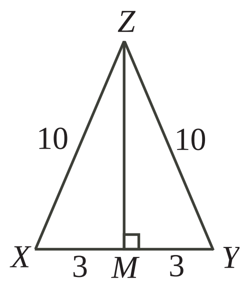
Since $XY = 6$, then the area of $\triangle CXY$ is $\frac{1}{2}(6)(3\sqrt{3}) = 9\sqrt{3}$.

Area of $\triangle XYZ$

Here, $XY = 6$ and $XZ = YZ = 10$.

Again, we drop an altitude from Z to XY.

Since $\triangle XYZ$ is isosceles, then this altitude meets XY at its midpoint, M.



Note that $XM = MY = \frac{1}{2}(XY) = 3$.

By the Pythagorean Theorem, $ZM = \sqrt{ZX^2 - XM^2} = \sqrt{10^2 - 3^2} = \sqrt{91}$.

Since $XY = 6$, then the area of $\triangle XYZ$ is $\frac{1}{2}(6)(\sqrt{91}) = 3\sqrt{91}$.

Therefore, the total surface area of solid $CXYZ$ is $24+24+9\sqrt{3}+3\sqrt{91} = 48+9\sqrt{3}+3\sqrt{91}$.

(c) Step 1: Examination of $\triangle MDN$

We know that $DM = 4$, $DN = 2$, and $\angle MDN = 60°$ (because $\triangle EDF$ is equilateral).

Since $DM : DN = 2 : 1$ and the contained angle is 60°, then $\triangle MDN$ must be a 30°-60°-90° triangle.

Therefore, MN is perpendicular to DF.

Using the ratios in the special triangle, $MN = \sqrt{3}DN = 2\sqrt{3}$.

We could have instead calculated the length of MN using the cosine law to determine this.

Step 2: Calculation of CP

We know that $QC = 8$ and $\angle QCP = 60°$.

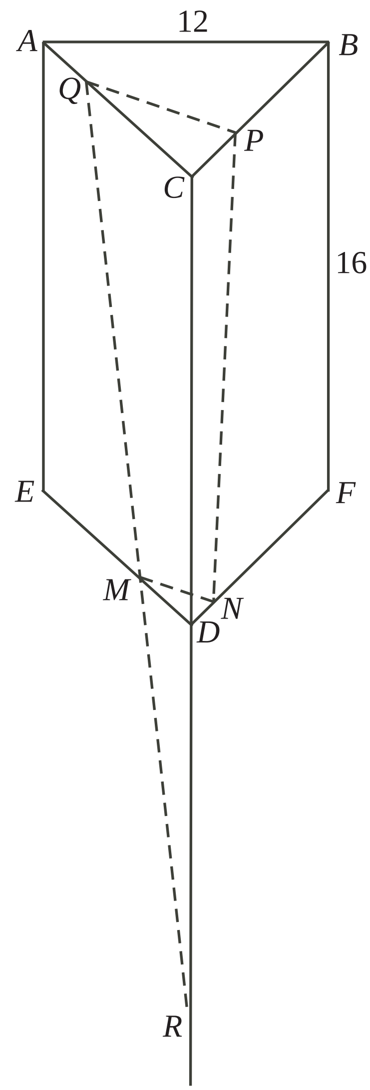
Since MN is perpendicular to DF, this tells us that the plane $MNPQ$ is perpendicular to the plane $BCDF$.

Since QP is parallel to MN (they lie in the same plane $MNPQ$ and in parallel planes ACB and DEF), then QP is perpendicular to CB.

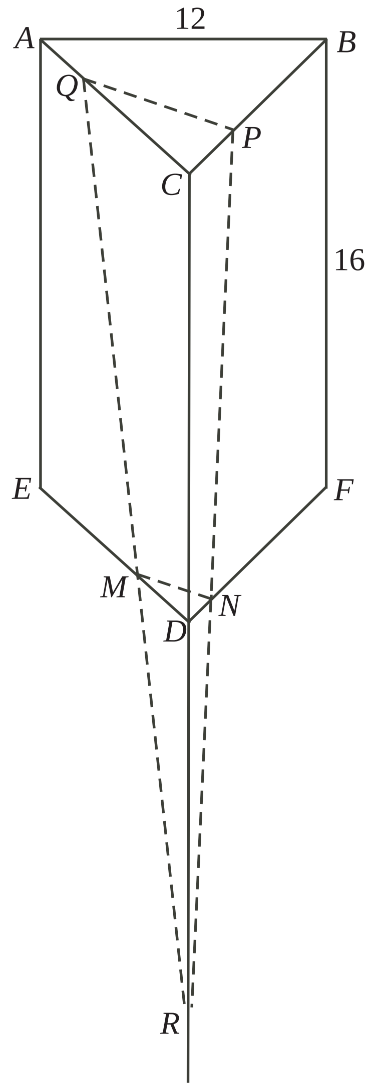
Therefore, $\triangle QCP$ is right-angled at P and contains a 60° angle, making it also a

30°-60°-90° triangle.
Thus, $CP = \frac{1}{2}(CQ) = \frac{1}{2}(8) = 4$ and $QP = \sqrt{3}CP = 4\sqrt{3}$.

Step 3: Construction
Extend CD downwards.
Next, extend QM until it intersects the extension of CD at R. (Note here that the line through QM will intersect the line through CD since they are two non-parallel lines lying in the same plane.)



Consider $\triangle RDM$ and $\triangle RCQ$.
The two triangles share a common angle at R and each is right-angled ($\triangle RDM$ at D and $\triangle RCQ$ at C), so the two triangles are similar.
Since $QC = 8$ and $MD = 4$, then their ratio of similarity is $2 : 1$.
This means that $RC = 2RD$, ie. D is the midpoint of RC.
Since $CD = 16$, then $DR = 16$.
Similarly, since $CP : DN = 2 : 1$, then when PN is extended to meet the extension of CD, it will do so at the same point R.



Step 4: Calculation of volume of $QPCDMN$

The volume of $QPCDMN$ equals the difference between the volume of the triangular based pyramid $RCQP$ and the volume of the triangular based pyramid $RDMN$.
(Another name for a triangular based pyramid is a tetrahedron.)
The volume of a tetrahedron equals one-third times the area of the base time the height.
The area of $\triangle CPQ$ is $\frac{1}{2}(CP)(QP) = \frac{1}{2}(4)(4\sqrt{3}) = 8\sqrt{3}$.
The area of $\triangle DNM$ is $\frac{1}{2}(DN)(MN) = \frac{1}{2}(2)(2\sqrt{3}) = 2\sqrt{3}$.
The length of RD is 16 and the length of RC is 32.
Therefore, the volume of $QPCDMN$ is $\frac{1}{3}(8\sqrt{3})(32) - \frac{1}{3}(2\sqrt{3})(16) = \frac{256\sqrt{3}}{3} - \frac{32\sqrt{3}}{3} = \frac{224\sqrt{3}}{3}$.